ELEMENTARY AND SECONDARY SCHOOL TRAINING IN MATHEMATICS*

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1. Introduction. The general problem to which these comments pertain is that of adapting our public schools to the needs of society and the nation. Attention is restricted, however, primarily to the question of mathematical instruction, the field in which the writer is best qualified to offer advice. This is also a phase of the general problem to which a peculiar importance attaches as a result of the national emergency and the associated critical shortage of scientifically trained manpower. The shortage is expected to become worse before it gets better, since industry, the government, and the military services are making increasingly heavy demands with no signs of a corresponding increase in the supply. A substantial improvement in the situation could be effected by remedying some of the serious defects in elementary and secondary school mathematical training. It is the present object to support this assertion by discussing such defects and suggesting remedial measures.

2. The principal questions. Recent discussions of educational problems have raised subsidiary questions which becloud some of the main issues and present a danger to effective progress. These troublesome and almost irrelevant problems include (1) whether certain subjects are better taught now than at some previous time (2) who is to blame for some of the recognized shortcomings of our schools and (3) whether we are preparing most students to meet their expected needs in later life. We should rather concentrate on the magnitude and nature of our national needs, on the obstacles to meeting them and on methods for overcoming these obstacles.

3. Mathematical shortcomings of our schools. To commence with generalizations, our high schools are sadly deficient both in preparing students for college and in offering adequate education to those not bound for college. Our elementary schools, in turn, are deficient in preparing students for high school.

Children of average to superior abilities, in the earliest grades, are frequently (perhaps generally) offered no encouragement to proceed at their natural pace in learning those aspects of arithmetic which appeal to them. At a slightly later stage, they are introduced to the fundamental operations of addition, subtraction, multiplication and division, but they are generally not drilled in such basic necessities as the multiplication tables. As a consequence they enter high school severely handicapped, save for that small proportion who learn so readily that they need no drill. In high school, the mathematics courses hit a slow pace, partly because the students have inadequate backgrounds, partly because there is no genuine incentive for the schools to provide suitable courses

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or for the students to take them. As a consequence, the colleges lose a year or two of mathematical training by having to teach to the freshmen, and frequently to the sophomores, courses which properly belong in the high schools. For students in the humanities, this situation is less serious than for those in the sciences and engineering. The latter are delayed to such an extent in commencing their essential studies that (1) a net loss of at least a year is difficult to avoid before their training is complete and (2) their programs are so crowded as to preclude many of the broadening studies which should form part of their general education. The delay carries over into the graduate schools, where we find ourselves teaching large proportions of our students material which belongs in a good undergraduate program.

Before substantiating the foregoing remarks, let it be noted that exceptional schools exist, generally in certain urban areas, and that in other schools exceptional teachers can be found who somehow partially counteract the general difficulties.

While a wealth of data could be offered in support of the preceding statement of shortcomings, a selection will be made, for convenience and brevity, from experience with two categories of students at the University of Illinois: (1) students who enter the College of Liberal Arts and Sciences with deficiencies in mathematics and (2) students in the elementary schools program of the College of Education. There were 234 students in the first group in the period 1949 to 1951 and 268 in the second between 1947 and 1952. Deficiencies in mathematics imply only failure to have taken one full year each of high school algebra and plane geometry. The lack may be due to lack of opportunity or to a deliberate avoidance of the subject for one reason or another. There is evidence that a good proportion of the students with such deficiencies (constituting about 10% of the L. A. S. freshmen) are suitable, though poorly prepared, college material. Identical standardized arithmetic tests have been administered to both groups. The results reveal a shocking inability to handle elementary arithmetic. To mention a few examples from data supplied by Mr. Clarence Phillips of the Mathematics Department, only 41% of the first group and 59% of the second correctly figured one year's interest at 6% on $175; the percentages of success were 34 and 55 in computing $7-6+2-4$, and 30% and 53% in arranging the numbers .40, 2.5 and .875 in order of magnitude. The difference between the two groups is due to the fact that the second group (1) had more mathematics in high school, (2) is more selective as to admission and (3) contains 77% seniors and graduate students, while the first group is almost all freshmen.

Of the students entering with mathematical deficiencies 50% fail to attain sophomore standing. This percentage is far out of line with the native abilities within the group and reveals the handicap of a student who is so poorly prepared by his high school.

The College of Education students just mentioned are required to take an arithmetic course in the Mathematics Department, intended to deepen their

understanding of what they will soon be teaching. Many of them are deplorably weak on the fundamentals, will hesitate over such things as eight times seven or seven plus six and will frankly express their easily understood fear of teaching arithmetic. This often takes place in the second semester of the senior year, after they have done practice teaching and a few months before they will be on the job, perhaps unconsciously transmitting their own aversions and lack of confidence to their students.

Turning to the College of Engineering, suffice it to remark that the inadequate mathematical preparation of the entering freshmen recently led to the establishment of a joint committee from that College, the College of Education and the Department of Mathematics. The work of this committee culminated in a pamphlet entitled *Mathematical Needs of Prospective Students at the College of Engineering of the University of Illinois*, which has been widely circulated among Illinois high schools. At present, a similarly composed committee, under the chairmanship of one of our University High School mathematics teachers, is studying means of adjusting the high school program to meet these needs. This cooperative effort is encouraging. It is to be hoped that the work of the committee will lead to widespread improvements in the teaching of mathematics and will serve as a model for cooperation elsewhere.

4. Underlying causes for shortcomings. This partly speculative section could run to great lengths. To avoid that, a few false principles will be listed, with brief comments and with no effort to estimate how widely these principles are accepted by those responsible for administering our schools.

a. *The theory that drill and deliberate memorizing must be avoided, especially in the lower grades.* This clearly works to the detriment of (1) learning the multiplication tables, (2) learning the alphabet at the proper time, (3) learning to spell, (4) learning the essentials of English grammar, (5) at a somewhat later stage developing a vocabulary when studying a foreign language and (6) acquiring the study habits demanded by effective college work. This theory is generally associated with the unwarranted belief that techniques will be incidentally acquired.

b. *The theory that local needs and desires should dominate in determining curricula, to the practical exclusion of needs on a national scale.*

c. *The theory that high schools should limit their programs to those skills and manipulations that some group of individuals finds necessary to the average adult.*

d. *The belief that the less successful students should be kept in the same classes with the more successful.*

e. *The aversion to competition among students.* This, and the previous item have a deadening effect on those who should be stimulated and encouraged.

5. A proposed guiding principle. To quote from a letter by Professor J. W.

Peters of the Department of Mathematics at the University of Illinois, "The American public high school has the responsibility to develop and administer a sound educational program which will provide for the education at the high school level of every youth to the full extent of his capacity and ability." This is a lofty ideal, but one which we can take as a guide, even though its full realization may be in the distant future.

6. Possible remedial measures. The following suggestions were offered for the consideration of the School Problems Commission.

a. The establishment of standards on a state-wide basis for the grade schools and high schools. The enforcement of such standards would require some sort of testing procedures. The merits of the Regents Examinations of the State of New York might be considered in this connection. In the establishment of standards and indeed in all phases of studying school problems, it is essential that due consideration be given to the views of scholars and scientists as well as to those of Departments and Colleges of Education, parents, industrial employers and educational administrators.

b. The establishment of adequate college entrance requirements. This would involve upward revisions, which could be introduced only gradually, as elementary and secondary schools adapt themselves thereto. In this connection, certain quotations may be in order from Bulletin Number 9 of the Illinois Curriculum Program series, entitled *New College Admission Requirements Recommended*, issued through the office of the State Superintendent of Public Instruction.

"The specification by the colleges of certain high school courses to be taken by all students seeking college entrance sets definite limitations to curriculum revision. If a considerable block of courses must be retained in the high school to provide for the preparation of students who hope to go to college, the opportunity to re-examine the total high school curriculum and to replan the program in terms of the needs of all high school youth is hereby curtailed." (p. 5.)

This disturbing quotation suggests that no essential college preparatory courses are recognized on the basis of all our educational experience to date.

"The committee recognizes that small high schools will not always be able to provide a sufficient variety of specialized courses to meet the need for the special programs of all its graduates. In such cases, the colleges are urged to make provision for the basic specialized work with as little handicap to the student as possible." (p. 14.)

This implies that the colleges are to continue, as a matter of policy, and perhaps even to expand, their offerings of high school types of instruction.

"With limited resources, the high school's first responsibility is to provide education of general value to all its students, rather than to provide for the

specialized needs of parts of the student body when the latter effort is taken at the expense of a good program of general education." (p. 13.)

This is at variance with the guiding principle suggested above. It implies that, although talent is uniformly distributed throughout the population, students in certain communities are to be denied the opportunity for their full development.

ON SEQUENCES OF OPERATIONS IN COMPLETE VECTOR SPACES

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1. Introduction. The first purpose of this paper is to describe how various results of classical analysis can be summarized in two general theorems concerning linear operations from a Banach space into a normed vector space. The two theorems are known as the *principle of uniform boundedness* and the *principle of condensation of singularities*. Our second object is to show how one can generalize these principles in the case of certain sequences of non-linear operations. In order to formulate our principles we start from simple examples, namely the summation of infinite sequences and the divergence problem of Fourier series of continuous functions.

2. The consistency of summation methods. As is well known, a real matrix-summation correlates with any sequence of real numbers $\{x_n\}$ a new sequence $\{u_m\}$ by means of a fixed infinite square matrix (a_{mn}) $(m, n=1, 2, \cdots)$; more precisely

$$u_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + \cdots \qquad (m = 1, 2, \cdots), \tag{1}$$

where the infinite series on the right hand side must be considered as a formal expression. It may diverge if the sequence $\{x_n\}$ is not sufficiently regular. However, it is natural to require of a summation method at least that it be efficient in the case of any convergent sequence, *i.e.* if $x_n \to \xi$ then all formal series (1) must converge and also $u_m \to \xi$ as $m \to \infty$. If a matrix-summation method (a_{mn}) has this basic property it is called a *consistent* or *regular* summation process. Perhaps the simplest non-trivial example of such consistent methods is the summation by arithmetical means.

There is a simple necessary and sufficient condition in order that a matrix-summation method should be consistent. Namely, according to a theorem of H. Steinhaus and O. Toeplitz [1], a summation process (a_{mn}) is regular if and only if

(1) *all series* $\sum_n a_{mn}$ *converge and their sequence tends to* 1 *as* $m \to \infty$;
(2) $\sum_n |a_{mn}| \leq H < \infty$, *where* $H > 0$ *is independent of* m;
(3) *for every fixed value of* n, $a_{mn} \to 0$ *as* $m \to \infty$.

In the case of summation by the first or higher arithmetical means one can easily see that the above conditions are all satisfied.

The sufficiency of the conditions (1)–(3) can easily be proved using everyday tricks of analysis. The necessity of conditions (1) and (3) is evident: If (a_{mn}) is a consistent method we must have $u_m \to 1$ for the convergent sequence $(1, 1, 1, \cdots)$, that is to say $u_m = \sum_n a_{mn} \to 1$ as $m \to \infty$. Similarly u_m must tend to zero in the case of $(0, 0, \cdots, 0, 1, 0, 0, \cdots)$, in other words $u_m = a_{mn} \to 0$ as $m \to \infty$. The necessity of condition (2) causes more difficulties in the proof, but, as a matter of fact, this condition is the crux of the Steinhaus-Toeplitz theorem, and it is worthwhile to investigate it more closely.

Before, and not too long after the discovery of this theorem, other results were found in various parts of analysis each of them involving certain conditions which are very similar to (2). The first of these results is the famous theorem of H. Hahn [2] concerning interpolations, then followed Pólya's theorem on the convergence of various quadrature processes [3], H. Hahn's and I. Schur's theorems about the singular integrals and summation processes (respectively [4], [5]) and other results.

It becomes obvious very soon that all these conditions must be special cases of some very general rule, and, indeed, in 1927 St. Banach and H. Steinhaus published a theorem [6] which includes all former results of the above type. This theorem is commonly known today as the principle of uniform boundedness of linear operations or the Banach-Steinhaus theorem.

3. Linear operations in vector spaces. It is evident that the formulation of such a summarizing result requires the introduction of new mathematical concepts. In our case the discovery of the importance of *complete vector spaces* played a major rôle. A *vector space* E is a set of elements (denoted by x, y, z, *etc.*) for which two basic operations are defined: the *addition* of two arbitrary elements and the *multiplication* of an arbitrary element by real numbers. Thus $x+y$ and λx, with λ real, are defined as elements of E. It is natural to require $1 \cdot x = x$, moreover it is assumed that these two operations satisfy the usual laws of arithmetic (commutativity and associativity of the two operations and the distributive law). Then it follows immediately that a *zero element* $\theta \in E$ exists, which has the property that $x+\theta = x$ and $0 \cdot x = \theta$ for every $x \in E$.

Let us suppose that with every element x of the vector space E we have associated a non-negative number, which we shall call the *norm* of x and denote by $\|x\|$. We require that the norm should always be positive, except in the case of the zero element θ for which $\|\theta\| = 0$, and that it should satisfy the triangle inequality, *i.e.* $\|x+y\| \leq \|x\| + \|y\|$ for every $x, y \in E$, and finally that $\|\lambda x\| = |\lambda| \cdot \|x\|$ for every real λ. If a vector space E has these properties we say

that the space is *normed*. In this way our space E becomes a *metric space* because we can define the *distance* of any two elements $x, y \in E$ as $d(x, y) = \|x-y\|$; obviously this function $d(x, y)$ has the usual properties of a distance function. Now we can introduce a kind of analysis in the space, and we may speak about *convergent sequences*, *infinite series*, *limit elements* and so on.

We say that a sequence of elements $x_n \in E$ converges to ξ if $\|x_n-\xi\| \to 0$ as $n \to \infty$. The triangle inequality implies that a convergent sequence always satisfies the *Cauchy convergence criterion*, that is $\|x_m-x_n\| \to 0$ as $m, n \to \infty$ simultaneously. However the converse is not true in general, because there are vector spaces where the Cauchy condition is satisfied for some sequence $\{x_n\}$ but no element $\xi \in E$ exists such that $\|x_n-\xi\| \to 0$ $(n \to \infty)$ would be true. If a normed vector space has the property that the Cauchy condition always implies the existence of a limit element ξ, then it is called a *complete vector space* or a *Banach space*. The importance of these spaces was discovered almost simultaneously by St. Banach [7], H. Hahn [4], T. H. Hildebrandt [8], and N. Wiener [9], but Banach was the one who developed their theory in detail.

Let us consider now two normed vector spaces E and E'. If a mapping of E into E' is given, *i.e.*, if to every element x of E there corresponds an element $u = u(x)$ of the second space E', we speak about an *operation* $u(x)$ from E into E', and we say that $u(x)$ is the image of the element x. The operation $u(x)$ is called *additive* if $u(x+y) = u(x)+u(y)$ for every couple x, y. Another important concept is the boundedness of an operation: $u(x)$ is *bounded* if there exists a positive constant M such that $\|u(x)\| \leqq M \cdot \|x\|$ for every $x \in E$. It is easy to see that an additive and bounded operation satisfies the relation $u(\lambda x) = \lambda \cdot u(x)$ for every real λ and $x \in E$, *i.e.*, one may "multiply out" with a numerical factor λ. This is the reason that additive and bounded operations are called *linear*. It follows from the boundedness condition that $\text{l.u.b.}_{\|x\|=1} \|u(x)\|$ is finite, and one can see immediately that

$$\text{l.u.b.}_{\|x\|=1} \|u(x)\| = \text{l.u.b.}_{\|x\| \leqq 1} \|u(x)\| = \text{l.u.b.}_{x \neq 0} \frac{\|u(x)\|}{\|x\|}.$$

This important number is the *norm of the operation* $u(x)$, usually denoted by $|u|$.

4. The principle of uniform boundedness. Now we are able to state the first of our principles in the generality in which it was proved by Banach and Steinhaus: *Consider a sequence of linear operations $u_m(x)$ $(m = 1, 2, \cdots)$ from a fixed complete vector space E into one common or into different normed vector spaces. If the sequence $\{u_m(x)\}$ is such that* $\limsup_{m \to \infty} \|u_m(x)\| < \infty$ *for every element $x \in E$, i.e. if the image points of each fixed point x remain in a finite neighborhood of the origin, then $|u_m| \leqq H \leqq \infty$ $(m = 1, 2, \cdots)$, that is to say the sequence of the norms of the operations is bounded.*

We can easily prove this theorem by contradiction: Let us suppose that all

the conditions are satisfied, but the conclusion is not true, *i.e.*,

$$\limsup_{m\to\infty} |u_m| = +\infty. \tag{2}$$

Then we shall construct an element $x\in E$ such that $\limsup_{m\to\infty} \|u_m(x)\| = +\infty$, which would contradict one of our hypotheses. To do so, we first notice that for every operation $u_m(x)$ there exists at least one element $x_m\in E$ for which

$$|u_m| \geqq \|u_m(x_m)\| \geqq \tfrac{1}{2}|u_m| \quad \text{and} \quad \|x_m\| = 1. \tag{3}$$

This follows immediately from the definition of the norm of the operation $u_m(x)$.

To start the construction, we notice that according to (2) and (3) we can choose an index $m_1\geqq 1$ such that $\|u_{m_1}(x_{m_1})\| \geqq \frac{1}{2}|u_{m_1}| \geqq 1$. Let us suppose that the indices $m_1<m_2<\cdots<m_{i-1}$ $(i\geqq 2)$ have already been determined in a convenient way. We shall now define m_i: first let

$$y_i = x_{m_1} + \frac{x_{m_2}}{2|u_{m_1}|} + \cdots + \frac{x_{m_{i-1}}}{2^{i-2}|u_{m_{i-2}}|}.$$

Since the conditions of the principle are supposed to be true we have

$$\|u_m(y_i)\| \leqq H(y_i) < +\infty \qquad (m = 1, 2, \cdots), \tag{4}$$

for a sufficiently large value of $H(y_i)>0$. Now we choose the index $m_i>m_{i-1}$ so large that

$$\|u_{m_i}(x_{m_i})\| \geqq 2^{i-1}\cdot |u_{m_{i-1}}| \cdot (H(y_i) + i + 1). \tag{5}$$

Since the right hand side is independent of m_i, this choice is possible by (2) and (3). In this way we obtain a subsequence $\{u_{m_i}(x)\}$ $(i=1, 2, \cdots)$ which satisfies the relations (4) and (5).

Finally we define x in the form of an infinite series

$$x = x_{m_1} + \frac{x_{m_2}}{2|u_{m_1}|} + \cdots + \frac{x_{m_i}}{2^{i-1}|u_{m_{i-1}}|} + \cdots.$$

The partial sums of this series were denoted earlier by y_i. It will be convenient to abbreviate also the remainder terms writing

$$z_i = \frac{x_{m_{i+1}}}{2^i\cdot|u_{m_i}|} + \frac{x_{m_{i+2}}}{2^{i+1}\cdot|u_{m_{i+1}}|} + \cdots, \qquad (i = 1, 2, \cdots).$$

Thus we have for every $i\geqq 2$

$$x = y_i + \frac{x_{m_i}}{2^{i-1}\cdot|u_{m_{i-1}}|} + z_i.$$

The convergence of these series is evident, because the space E is complete and the Cauchy criterion is satisfied. More precisely, using $\|x_i\| = 1$ and $\|u_{m_i}\| > 1$ we can deduce the inequality $|u_{m_i}|\cdot\|z_i\| < 1$ $(i=2, 3, \cdots)$, and (3) and (5)

show that $|u_{m_i}| \to \infty$ as $i \to \infty$.

Now it is easy to prove that lim $\sup_{m\to\infty} \|u_m(x)\| = +\infty$ which contradicts the hypotheses of our principle. For, we obtain from (6)

$$\|u_{m_i}(x)\| = \left\|u_{m_i}(y_i) + u_{m_i}\left(\frac{x_{m_i}}{2^{i-1}\cdot|u_{m_{i-1}}|}\right) + u_{m_i}(z_i)\right\|$$

$$\geqq \frac{\|u_{m_i}(x_{m_i})\|}{2^{i-1}\cdot|u_{m_{i-1}}|} - \|u_{m_i}(y_i)\| - \|u_{m_i}(z_i)\|.$$

Hence using (4), (5) and the inequality $\|u_{m_i}(z_i)\| \leqq |u_{m_i}| \cdot \|z_i\| < 1$ we get

$$\|u_{m_i}(x)\| > (H(y_i) + i + 1) - H(y_i) - 1 = i$$

for every $i = 1, 2, \cdots$. Thus, indeed lim $\sup_{m\to\infty} \|u_m(x)\| = +\infty$, which completes the proof of the principle of uniform boundedness.

5. Examples and applications. Let us now return to the consistency question of matrix-summation methods. As the first application of our principle we shall prove the necessity of condition (2) of the Steinhaus-Toeplitz theorem: Let E be the set of all convergent real sequences $\{x_n\}$ which we shall from now on denote by small Roman letters (x, y, *etc.*) instead of $\{x_n\}$, $\{y_n\}$, *etc.* We define the sum of two arbitrary sequences and the product of a sequence and a real number in the usual way as $x+y = \{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $\lambda x = \lambda \cdot \{x_n\} = \{\lambda x_n\}$, respectively. Obviously $x+y \in E$ and $\lambda x \in E$, and one can also see that these operations satisfy the usual requirements. Thus the set of all convergent real sequences forms a vector space E. The space will be normed if we introduce the norm $\|x\| = \|\{x_n\}\| = \text{l.u.b.}_{(n)} |x_n|$. This norm is finite because the sequence x is supposed to be convergent. It is easy to verify that this normed vector space is complete.

A matrix-summation (a_{mn}) can now be considered as a sequence of linear operations

$$u_m(x) = u_m(\{x_n\}) = a_{m1}x_1 + a_{m2}x_2 + \cdots, \quad (m = 1, 2, \cdots),$$

from the space E into the space of all real numbers R. Let us suppose that $\{u_m(x)\}$ $(m \to \infty)$ converges to the same limit as the arbitrary convergent sequence $\{x_n\}$. Hence the sequence $\{u_m(x)\}$ is *a priori* bounded for every element $x = \{x_n\} \in E$, so that the main condition of the principle of uniform boundedness is satisfied. Since E is complete we can apply the principle and it follows that the sequence of the norms $\{|u_m|\}$ is bounded.

Let us now evaluate the norm of $u_m(x)$, and thus obtain a necessary condition for the regularity of the summation process (a_{mn}). Since the sequence

$$y_n = (\text{sign } a_{m1}, \text{ sign } a_{m2}, \cdots, \text{ sign } a_{mn}, 0, 0, \cdots) \to 0$$

belongs to E and has norm 1 we see that

$$|u_m| \geqq \|u_m(y_n)\| = |a_{m1}| + |a_{m2}| + \cdots + |a_{mn}|.$$

$n \geqq 1$ being arbitrary it follows that $|u_m| \geqq \sum_{n=1} |a_{mn}|$. On the other hand it is obvious that

$$\|u_m(x)\| = \left| \sum_{n=1}^{\infty} a_{mn} x_n \right| \leqq \sum_{n=1}^{\infty} |a_{mn}|,$$

provided $|x_n| \leqq 1$ for $n=1, 2, \cdots$, *i.e.*, if $\|x\| \leqq 1$. Finally we see that $|u_m| = \sum_{n=1}^{\infty} |a_{mn}|$, and we have the necessity of condition (2).

As a second example we consider the divergence problem of the Fourier series of continuous functions at a given point. It has been known for quite a long time that there exist periodic functions with period 2π which are continuous everywhere, but whose Fourier series diverge in a prescribed point. Of course one can assume that the prescribed point is the origin $t=0$. The first example of such functions was given by P. du Bois Reymond [10], and the simplest known construction is due to L. Fejér [11]. H. Lebesgue [12] gave a proof of the existence of such functions which is very remarkable from our point of view. For what Lebesgue proves is nothing but the converse of the principle of uniform boundedness for the special case of Fourier series. It is unfortunate that Lebesgue did not investigate in what generality his method of proof works, for then the principle of uniform boundedness would have been known much earlier.

Having proved the general principle we can go on and use it to show the existence of periodic continuous functions with period 2π whose Fourier series diverge at the origin. For, the m-th partial sum of the Fourier series of $x(t)$ can be written in the form of a Dirichlet integral, and if $t=0$

$$s_m(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t) D_m(t) dt,$$

where for simplicity $D_m(t) = (\sin (m+\frac{1}{2})t)/\sin \frac{1}{2}t$. (See for example A. Zygmund's textbook on trigonometrical series, pp. 20–21.) Evidently the set of all 2π-periodic and continuous functions $x(t)$ forms a vector space E which can be normed by $\|x\| = \max |x(t)|$. It is easy to see that E is complete. Hence we can consider the value of the m-th partial sum at $t=0$ as a linear operation $s_m(x)$ from the space E into the space of all real numbers R. If $\{s_m(x)\}$ $(m \to \infty)$ were convergent for every element $x=x(t) \in E$, then according to the principle of uniform boundedness the sequence $\{|s_m|\}$ would be bounded. However we shall immediately see that this is not the case, so that there exist continuous functions whose Fourier series diverge at the origin.

It is obvious that

$$|s_m| \leqq L_m = \frac{1}{2\pi} \int_0^{2\pi} |D_m(t)| \, dt.$$

On the other hand, let $\epsilon<\pi/(2m+1)$ and consider the 2π-periodic and continuous function

$$x_\epsilon(t) = \begin{cases} \text{sign } D_m(t) & \text{for } \dfrac{2k}{2m+1} + \epsilon \leqq t \leqq \dfrac{2(k+1)}{2m+1} - \epsilon \\ \text{linear otherwise} & (k = 0, \pm 1, \pm 2, \cdots). \end{cases}$$

We have

$$\|s_m(x_\epsilon)\| \geqq \frac{1}{2\pi}\int_0^{2\pi} |D_m(t)|\,dt - \frac{1}{\pi}\sum_{k=0}^{2m}\int_{(2k\pi)/(2m+1)-\epsilon}^{(2k\pi)/(2m+1)+\epsilon} |D_m(t)|\,dt$$

$$\geqq L_m - \frac{2\epsilon}{\pi}(2m+1)^2,$$

because $|D_m(t)| = |1+2\cos t+\cdots+2\cos mt| \leqq 2m+1$. Since $\epsilon>0$ is arbitrarily small, we obtain $|s_m| \geqq \lim_{\epsilon\to 0}\|s_m(x_\epsilon)\| = L_m$ and finally the norm of the operation $s_m(x)$ turns out to be $|s_m| = L_m$. This quantity is called the m-th *Lebesgue constant* and it plays an important rôle in the theory of Fourier series.

A rough estimate is now sufficient to prove that $|s_m| \to \infty$: for, we have

$$L_m > \frac{1}{2\pi}\sum_{k=0}^{2m}\int_{\alpha_k}^{\beta_k} \frac{|\sin(m+\frac{1}{2})t|}{|\sin\frac{1}{2}t|}\,dt \geqq \frac{\sqrt{2}}{4\pi}\sum_{k=0}^{2m}\int_{\alpha_k}^{\beta_k}\frac{1}{|\sin\frac{1}{2}t|}\,dt,$$

where for the sake of simplicity $\alpha_k=(4k+1)\pi/(4m+2)$ and $\beta_k=(4k+3)\pi/(4m+2)$, because $\sin t \geqq 1/\sqrt{2}$ in the interval $(\pi/4, 3\pi/4)$. Using $|\sin t| \leqq t$ $(t\geqq 0)$ it follows that

$$L_m > \frac{\sqrt{2}}{\pi}\sum_{k=0}^{2m}\frac{1}{4k+3}.$$

This last inequality clearly shows that $L_m\to\infty$ as $m\to\infty$. Thus there exist continuous functions whose Fourier series diverge at any prescribed point. We note briefly that a precise estimate for L_m due to F. Fejér [13], states $L_m = 4/\pi^2 \log m + O(1)$.

6. The principle of condensation of singularities. Instead of giving further applications of the principle of uniform boundedness (as for example the summability of Fourier series by arithmetic means), let us continue our investigations concerning the divergence problem of Fourier series. After the preceding considerations one may expect that the divergence property is much more general than the result discussed above about divergence in finitely many prescribed points. Indeed, P. du Bois Reymond [10] proved the following: *Let S be an arbitrary countable set of real numbers t $(0\leqq t\leqq 2\pi)$. Then there exists a periodic continuous function with period 2π such that its Fourier series diverges at every $t\in S$.* Thus, for example, we can construct continuous functions whose set of divergence points is everywhere dense.

The theorem can be proved in several ways. However all proofs have one thing in common: they all use the fact that there exist continuous functions with Fourier series diverging at a point (or what is the same, $L_m \to \infty$), but only very few of the other properties of Fourier series are used. These facts suggest that the theorem can be extended to a greater generality than the case of Fourier series. As a matter of fact, the method used in the proofs goes back at least to the time of Riemann (Habilitationsschr., 1854) and was later improved by several authors. H. Hankel [14] was the first one who considered the problem independently of its origin and tried to present the method and the results obtained by it in the greatest possible generality. He also introduced the name: *principle of condensation of singularities* (Condensation der Singularitäten).

The introduction of the concept of complete vector spaces was the step necessary for a better understanding. In fact, St. Banach and H. Steinhaus [6] succeeded in proving a general principle which includes the theorem on Fourier series and many other results and saves the application of the method in each particular case. It is the following: *Consider a double-sequence of linear operations* $u_{mn}(x)$ $(m, n = 1, 2, \cdots)$ *from a complete vector space* E *into arbitrary normed vector spaces* E'_{mn}. *Suppose that for every* $m \geqq 1$ *there exists an element* x_m *of the space* E *such that* $\limsup_{n\to\infty} \|u_{mn}(x_m)\| = +\infty$. *Then there exists a common element* $x \in E$ *such that* $\limsup_{n\to\infty} \|u_{mn}(x)\| = +\infty$ *for every* $m \geqq 1$.

The connection of this theorem with the divergence of Fourier series on enumerable sets is evident: Let $\{t_1, t_2, \cdots, t_m, \cdots\}$ be the set given in advance and let $s_{mn}(x)$ denote the n-th partial sum of the Fourier series of $x(t)$ in the point t_m. According to the principle of uniform boundedness there exists a continuous function $x_m(t)$ whose Fourier series diverges to infinity at the given point t_m. Thus the conditions of our second principle are satisfied and it follows that there exists a continuous function $x(t)$ such that all sequences $\{s_{mn}(x)\}$ (m fixed, $n \to \infty$) diverge, in other words the Fourier series of $x(t)$ diverges in all points t_m $(m \geqq 1)$.

This example shows clearly that in the applications one usually applies the two principles simultaneously. First, using the principle of uniform boundedness we show that the conditions of the principle of condensation are satisfied, and then we apply the second principle. This is the situation, for instance, in the case of the *Lagrange interpolation* which we shall investigate more closely. The proof of the principle itself is similar to the proof given in section 4, and for this reason we shall omit it.

7. The divergence of the Lagrange interpolation. For simplicity let us consider only the case of the equidistant interpolation of continuous functions $x(t)$ $(0 \leqq t \leqq 1)$. As is well known, we define the n-th Lagrange approximation of $x(t)$ as

$$l_n(x; t) = \sum_{k=0}^{n} x\left(\frac{k}{n}\right)\lambda_k^{(n)}(t), \qquad (n = 1, 2, \cdots),$$

where

$$\lambda_k^{(n)}(t) = (-1)^{n-k}\frac{n^n}{k!(n-k!)}\cdot t\left(t-\frac{1}{n}\right)\left(t-\frac{2}{n}\right)\cdots(t-1)\left(t-\frac{k}{n}\right)^{-1}.$$

Let C denote the space of all continuous functions $x(t)$, $0\leqq t\leqq 1$. If we introduce the norm $\|x\|=\|x(t)\|=\max|x(t)|$, then C becomes a complete vector space. Now we can consider $l_n(x;t)$ as a linear operation from C into C itself, or if we wish, we can also say that for every fixed value of t, $(0\leqq t\leqq 1)$, $l_n(x;t)$ is a linear operation from C into the space of real numbers R. Let us make the second choice, so that we obtain a sequence of operations $\{l_n(x;\tau)\}$ $(n\to\infty)$ depending on the parameter τ $(0\leqq\tau\leqq 1)$.

The norm of the operation $l_n(x;\tau)$ can be determined very easily: It is obvious that $|l_n(\tau)|\leqq\sum_0^n|\lambda_k(\tau)|$. On the other hand let $x_n=x_n(t)=\operatorname{sign}\lambda_k(t)$ for $t=k/n$, $(k=0, 1, \cdots, n)$, and linear otherwise. We have $x_n\in C$, $\|x_n\|=1$, and hence $|l_n(\tau)|\geqq\|l_n(x_n;\tau)\|=\sum_0^n|\lambda_k(\tau)|$. Thus it follows that $|l_n(\tau)|=\sum_0^n|\lambda_k(\tau)|$. We next investigate the boundedness of the sequence $\{|l_n(\tau)|\}$. It will turn out that $\limsup_{n\to\infty}|l_n(\tau)|=+\infty$ for every $0<\tau<1$. According to the principle of uniform boundedness therefore there exists a continuous function whose Lagrange interpolation diverges to infinity for any given τ, $0<\tau<1$.

Let us suppose that $\tau\neq\frac{1}{2}$ and $0<\tau<1$. In order to estimate $|l_n(\tau)|$ we choose the integer $a=a_n(\tau)$ so that $a/n<\tau<(a+1)/n$. If τ is a rational number this inequality gives some restriction also on the possible choice of $n\geqq 1$. However we shall get rid of this restriction immediately. According to the definition of $\lambda_k(\tau)$ we have

$$\begin{aligned}|\lambda_k(\tau)| &\geqq \frac{n^n}{k!(n-k)!}\cdot\frac{a}{n}\cdot\frac{a-1}{n}\cdot\cdots\cdot\frac{1}{n}\left(\tau-\frac{a}{n}\right)\left(\frac{a+1}{n}-\tau\right)\\ &\quad\cdot\frac{1}{n}\cdot\frac{2}{n}\cdot\cdots\cdot\frac{n-a-1}{n}\Big/\left|\tau-\frac{k}{n}\right|\\ &\geqq \frac{a!(n-a)!}{k!(n-k)!}\left(\tau-\frac{a}{n}\right)\left(\frac{a+1}{n}-\tau\right)\\ &\geqq \frac{1}{2n}\cdot\frac{a!(n-a)!}{k!(n-k)!}\operatorname*{minimum}_{0\leqq m\leqq n}\left|\tau-\frac{m}{n}\right|,\end{aligned}$$

because $|\tau-k/n|\leqq 1$, and $(n-a-1)!n\geqq(n-a)!$. Evidently this inequality holds for every choice of $n\geqq 1$. Using the relation $\sum_0^n\binom{n}{k}=2^n$ we obtain from the above inequality

$$|l_n(\tau)|=\sum_{k=0}^n|\lambda_k(\tau)|\geqq\frac{2^n}{2n}\cdot\frac{a!(n-a)!}{n!}\operatorname*{minimum}_{0\leqq m\leqq n}\left|\tau-\frac{m}{n}\right|.$$

Using the elementary inequalities $(\nu/e)^\nu < \nu! < 4\nu(\nu/e)^\nu$ we see that

$$|l_n(\tau)| \geqq \frac{2^n}{8n^2} \cdot \frac{a^a(n-a)^{n-a}}{n^n} \operatorname*{minimum}_{0\leqq m\leqq n} \left|\tau - \frac{m}{n}\right|$$

$$= \frac{1}{8n^2} (2\alpha(1-\alpha)^{1-\alpha})^n \operatorname*{minimum}_{0\leqq m\leqq n} \left|\tau - \frac{m}{n}\right|,$$

where $\alpha = \alpha_n(\tau) = a/n$. Since $\tau \neq \frac{1}{2}$ and $0 \leqq \tau - a/n = \tau - \alpha_n < 1/n$, we have $|\alpha_n - \frac{1}{2}| > \frac{1}{2}|\tau - \frac{1}{2}| > 0$, provided n is large enough. Moreover, since the convex function $x(t) = 2t^t(1-t)^{1-t}$, $0 \leqq t \leqq 1$, has its minimum at $t = \frac{1}{2}$ with $x(\frac{1}{2}) = 1$, we see that there is a constant $\eta = \eta(\tau) > 1$ such that $2\alpha_n^{\alpha_n}(1-\alpha_n)^{1-\alpha_n} \geqq \eta > 1$ holds for every sufficiently large $n = 1, 2, \cdots$. Therefore for every sufficiently large n

$$|l_n(\tau)| > \frac{\eta^n}{8n^2} \operatorname*{minimum}_{0\leqq m\leqq n} \left|\tau - \frac{m}{n}\right|; \qquad \eta = \eta(\tau) > 1.$$

To finish the proof we show that there are infinitely many n such that $\min_{(m)} |\tau - m/n| > 1/4n^2$. For, let us suppose that n is fixed, $n = p \geqq 2$, and $\min |\tau - m/p| = |\tau - m_1/p|$ and $\min |\tau - m/(p+1)| = |\tau - m_2/(p+1)|$. Since $0 < \tau < 1$ we have $1 \leqq m_1 \leqq p-1$ and $1 \leqq m_2 \leqq p$, provided p is large enough; $p \geqq n_0(\tau) \geqq 2$.

Since p and $p+1$ are relatively prime, it follows that $|m_1(p+1) - m_2 p| \geqq 1$. Consequently

$$\left|\tau - \frac{m_1}{p}\right| + \left|\tau - \frac{m_2}{p+1}\right| \geqq \left|\frac{m_1}{p} - \frac{m_2}{p+1}\right| \geqq \frac{1}{p(p+1)},$$

and so

$$\min_{(m)} \left|\tau - \frac{m}{p}\right| + \min_{(m)} \left|\tau - \frac{m}{p+1}\right| \geqq \frac{1}{2p^2}.$$

Hence for every $p \geqq 2$, $n = p$ or $n = p+1$ satisfies the required inequality, and so $|l_n(\tau)| \geqq \eta^n/32n^4$, $(\eta = \eta(\tau) > 1)$, is valid for infinitely many indices. Consequently $\limsup_{n\to\infty} |l_n(\tau)| = \infty$ $(\tau \neq 0, \frac{1}{2}, 1)$. If $\tau = \frac{1}{2}$ one must use a slightly different estimation and one can show that $|l_{2n+1}(\frac{1}{2})| \sim \log n/\pi$.

Using the principle of uniform boundedness we see that for every $0 < \tau < 1$ there exists an element $x_\tau \in C$ such that $\limsup_{n\to\infty} |l_n(x_\tau; \tau)| = +\infty$. Therefore we can apply the principle of condensation and obtain: Given an arbitrary enumerable set of abscissae t, $0 < t < 1$, there exists a continuous function whose equidistant Lagrange interpolation diverges at each point of the prescribed set. It is easy to see that at the points $t = 0$ and $t = 1$ the interpolation is convergent.

8. Generalizations of the principles. Although our principles are stated

in a general form which includes many theorems as special cases, it is nevertheless desirable to obtain further extensions. Let us consider first the principle of condensation: Using Fejér's example it is a simple matter to construct a continuous function $x(t)$ with Fourier series divergent in a non-denumerable set of points. It is conjectured that there exist continuous functions whose Fourier series diverge everywhere. In the case of the Lagrange interpolation the corresponding statement has already been proved by G. Grünwald [15]. However we have as yet been unable to clear up the situation completely. The most general form of the principle of condensation to date is due to W. Orlicz [16] and states: *Consider a sequence of linear operations* $u_n(x;\tau)$ *depending on a continuous parameter* $\tau(0\leqq\tau\leqq 1)$. *Suppose that the operations are continuous with respect to* τ. *Suppose moreover that for every* $\tau(0\leqq\tau\leqq 1)$ *there exists an* x_τ *such that* $\limsup_{n\to\infty}\|u_n(x_\tau;\tau)\|=+\infty$. *Then there exists also an element* x *so that* $\limsup_{n\to\infty}\|u_n(x;\tau)\|$ *holds for a non-denumerable (perfect) set of the values of* τ.

It is easier to generalize our principles in another direction, namely by assuming instead of linearity some less restrictive conditions on the operations $u_m(x)$ and $u_{mn}(x)$: If we investigate the original proof of Banach and Steinhaus we see that the principle of uniform boundedness can be extended to the case of bounded, homogeneous and subadditive (convex) operations. By definition these operations satisfy the conditions: $\|u(x)\|\leqq M\|x\|$, $u(\lambda x)=\lambda u(x)$ and $\|u(x+y)\|\leqq\|u(x)\|+\|u(y)\|$ for every choice of x, $y\in E$. It is easy to verify that these conditions are indeed less restrictive than linearity. We can go still somewhat further and replace homogeneity by the weaker condition $\|u(\lambda x)\|=|\lambda|\cdot\|u(x)\|$, but this is about the most general form of the principle one can obtain by the method of Banach and Steinhaus.

There is a recent result concerning almost everywhere approximation of measurable functions by polynomials [17] which is evidently closely related to the principle of uniform boundedness. However this last result does not follow from the form of the principle which is stated above, but suggests that the method of section 4 might be applied successfully in this case. This method can indeed be used to prove a still further extension of the principle [18]: A sequence of bounded operations $u_m(x)$ is called *asymptotically subadditive* if $\|u_m(\lambda x)\|=|\lambda|\cdot\|u_m(x)\|$ for every real λ, and

$$\|u_m(x+y)\|\leqq\|u_m(x)\|+O(|u_m|\cdot\|y\|)$$

uniformly for every x, y with $\|x\|$, $\|y\|\leqq 1$, and if also

$$\text{g.l.b.}_{\|y\|\leqq 1}\,[\|u_m(x+y)\|+\|u_m(x)\|-\|u_m(y)\|]\geqq o(|u_m|)$$

as $m\to\infty$, but not necessarily uniformly in $x\in E$. (In other words the left hand side of the last formula can be negative, but must be greater than $-c_m|u_m|$ where $c_m\to 0$ as $m\to\infty$. The O has its usual meaning, *i.e.* it denotes a function of $|u_m|$ and $\|y\|$ which in absolute value is less than $c|u_m|\cdot\|y\|$, where $c>0$ denotes a constant.)

The result is the following: If the sequence of operations $u_m(x)$ is asymptotically subadditive, then the principle of uniform boundedness is true. The situation is similar in the case of the principle of condensation [18]: If every single sequence $\{u_{mn}(x)\}$ (m fixed, $n \to \infty$) is asymptotically subadditive, then the principle can be applied.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, College of St. Thomas

The following results of the thirteenth William Lowell Putnam Mathematical Competition held March 23, 1953, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Norman Bauman, Marshall L. Freimer and Ralph M. Krause; to each of these a prize of forty dollars is awarded.

There was a tie for second place between The City College, New York, New York, and Cornell University, Ithaca, New York. A prize of two hundred and fifty dollars is awarded the Departments of Mathematics of each of these institutions. The members of the team from The City College, New York, New York, were Arnold M. Benson, Irwin Friedman and Samuel Jacob Klein; to each of these a prize of twenty-five dollars is awarded. The members of the team from Cornell University, Ithaca, New York, were Leonard Evens, David Hertzig and Daniel J. Kleitman; to each of these a prize of twenty-five dollars is awarded.

The fourth prize, one hundred dollars is awarded to the Department of Mathematics of the University of California, Berkeley, California. The members of the team were James Brown Herreshoff 4th, Eva Marianne Kallin and Harold Weitzner; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were Norman Bauman, Harvard University; Marshall L. Freimer, Harvard University; James Brown Herreshoff 4th, University of California (Berkeley); Samuel Jacob Klein, The City College (New York); Tai Tsun Wu, University of Minnesota. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, were George A. Baker, California Institute of Technology; William Morton Kahan, University of Toronto; Henry J. Landau, Harvard University; John S. Lew, Yale University; Richard Gordon Swan, Princeton University. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: Brooklyn College, Brooklyn, New York, the members of the team being Donald Solitar, Aaron Stein, Jack Towber; California Institute of Technology, Pasadena, California, the members of the team being George A. Baker, Robert D. Ryan, David F. Stevens; Columbia College, New York, New York, the members of the team being Lee Abramson, S. David Berkowitz, Elihu Lubkin; Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being Louis de Branges, Redmond O'Brien, Monroe Weinstein.

Fifteen individuals were given honorable mention. The names are listed in alphabetical order. Leonard E. Baum, Harvard University; Arnold M. Benson, The City College (N. Y.); David S. Berkowitz, Columbia College (N. Y.); Robert Hamilton Boyer, Carnegie Institute of Technology; Glen Eugene Bredon, Stanford University; Eugene Butkov, University of British Columbia; Louis de Branges, Massachusetts Institute of Technology; Wilfred Keith Hastings, University of Toronto; Daniel J. Kleitman, Cornell University; Ralph M. Krause, Harvard University; John Randolph Manning, Ursinus College (Collegeville, Pa.); Benjamin Muckenhaupt, Harvard University; Gion Carlo Rota, Princeton University; Donald Solitar, Brooklyn College; Harold Weitzner, University of California (Berkeley).

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: Arizona State College, Brooklyn College, Brown University, California Institute of Technology, Carnegie Institute of Technology, Carleton College, Catholic University of America, City College (N. Y.), Coe College, College of St. Thomas, Columbia College (N. Y.), Cooper Union, Cornell University, Harvard University, Haverford College, Holy Cross College, Iowa State College, Kenyon College, Knox College, La-Salle College, Laval University, Lynchburg College, Massachusetts Institute of Technology, McGill University, McMaster University, Memphis State College, New York University, Oregon State College, Polytechnic Institute of Brooklyn, Princeton University, Queen's University, Radcliffe College, Rosemont College, Rutgers University, Saint Joseph College, Stanford University, Syracuse University, U. S. Naval Academy, University of British Columbia, University of California (Berkeley), University of Detroit, University of Illinois, University of Miami, University of Minnesota, University of Rochester, University of Toronto, Ursinus College, Washington University (Saint Louis), Yale University,

The following additional colleges and universities entered individual contestants only: Barat College, Boston University, Clemson College, Hartwick College, Lincoln Memorial University, Loyola College, Ohio State University, Pomona College, Purdue University, University of Arizona, University of Kentucky, University of Omaha, University of Washington (Seattle), University of Wisconsin.

A total of 256 undergraduates representing 63 institutions took part in the Competition.

The departments of mathematics of any of the competing institutions may obtain the rankings of their individual contestants (except that the relative rankings of the first five will not be given) by writing to the Director. These rankings may now be communicated to the individual contestants by their departments. Any other departments of mathematics may obtain individual rankings for the purpose of selecting graduate students.

Participants in the Competition were given the following lists of problems:

PART I

MORNING SESSION: 9:00 A.M. TO 12:00 NOON

Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate all answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.

Omit one question. You must indicate which question is omitted.

1. Prove that, for every positive integer n,

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}$$

is more than $\frac{2}{3}n\sqrt{n}$ and less than

$$\frac{4n+3}{6}\sqrt{n}.$$

2. Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color.
3. If x_1, x_2, x_3 are real numbers and the sum of any two is greater than the third, show that

$$\frac{2}{3}\sum_{i=1}^{3} x_i \sum_{i=1}^{3} x_i^2 > \sum_{i=1}^{3} x_i^3 + x_1x_2x_3.$$

4. From the identity

$$\int_0^{\pi/2} \log \sin 2x\,dx = \int_0^{\pi/2} \log \sin x\,dx + \int_0^{\pi/2} \log \cos x\,dx + \int_0^{\pi/2} \log 2\,dx,$$

deduce the value of

$$\int_0^{\pi/2} \log \sin x\,dx.$$

5. Let P be a point from which three distinct normals can be drawn to a parabola. Show that the sum of the angles which these three normals make with the axis exceeds by a multiple of π the angle which the line joining P to the focus makes with the axis.
6. Show that the sequence

$$\sqrt{7}, \sqrt{7-\sqrt{7}}, \sqrt{7-\sqrt{7+\sqrt{7}}}, \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}, \cdots$$

converges, and evaluate the limit.
7. Assuming that the roots of $x^3+px^2+qx+r=0$ are all real and positive, find

the relation between p, q and r which is a necessary and sufficient condition that the roots may be the cosines of the angles of a triangle.

PART II

AFTERNOON SESSION: 2:00 TO 5:00 P.M.

Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate all answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.

Omit one question. You must indicate which question is omitted.

1. Is the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^{(n+1)/n}}$$
convergent? Prove your statement.
2. Let $a_0, a_1, \cdots, a_n$ be real numbers and let $f(x)=a_0+a_1x+\cdots+a_nx^n$. Suppose that, for every integer i, $f(i)$ is an integer. Prove that $n!\cdot a_k$ is an integer for each k.
3. Solve the equations
$$\frac{dy}{dx}=z(y+z)^n \qquad \frac{dz}{dx}=y(y+z)^n,$$
given the initial conditions $y=1$ and $z=0$ when $x=0$.
4. Determine the equation of a surface in three dimensional cartesian space which has the following properties: (a) it passes through the point (1, 1, 1); and (b) if the tangent plane be drawn at any point P, and A, B and C are the intersections of this plane with the x, y and z axes respectively, then P is the orthocenter (intersection of the altitudes) of the triangle ABC.
5. Show that the roots of $x^4+ax^3+bx^2+cx+d=0$, if suitably numbered, satisfy the relation $r_1/r_2=r_3/r_4$, provided $a^2d=c^2\neq 0$.
6. P and Q are any points inside a circle (C) with center C, such that $CP=CQ$. Determine the location of a point Z on (C) such that $PZ=QZ$* shall be a minimum.
7. Let w be an irrational number of $0<w<1$. Prove that w has a unique convergent expansion of the form
$$w=\frac{1}{p_0}-\frac{1}{p_0p_1}+\frac{1}{p_0p_1p_2}-\frac{1}{p_0p_1p_2p_3}+\cdots,$$
where $p_0, p_1, p_2, \cdots$ are integers and $1\leq p_0<p_1<p_2<\cdots$. If $w=\frac{1}{2}\sqrt{2}$, find p_0, p_1, p_2.

* The examination committee intended that this should read $PZ+QZ$, but the problem was put before the contestants as shown above.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A NOTE ON THE SYLVESTER-FRANKE THEOREM

HARLEY FLANDERS, University of California, Berkeley

L. Tornheim [1] has recently given a proof of the Sylvester-Franke theorem based on elementary transformations. We shall present here another proof which is based on the Grassmann exterior algebra [2].

Let L be a linear space of dimension n over a field of scalars k. For each non-negative integer p, there is associated to L in an intrinsic manner a new vector space, denoted by $\Lambda^p L$, whose elements are called p-vectors. This space may be characterized as follows. There is a function f which carries ordered p-tuples of vectors of L into $\Lambda^p L$, *i.e.*,

$$(x_1, \cdots, x_p) \to f(x_1, \cdots, x_p) \in \Lambda^p L \qquad \text{for } x_i \in L,$$

such that:

(a) f is linear in each variable. In short, f is multilinear.

(b) f is alternating. This means that if π is any permutation of the indices $1, 2, \cdots, p$, then

$$f(x_{\pi(1)}, \cdots, x_{\pi(p)}) = (\operatorname{sgn} \pi) f(x_1, \cdots, x_p)$$

for all elements $x_1, \cdots, x_p$ of L. Here $\operatorname{sgn} \pi$ denotes the sign of the permutation π. It is 1 if π is even and -1 if π is odd.

(c) The collection of all images $f(x_1, \cdots, x_p)$ of p-tuples of L generates the space $\Lambda^p L$.

(d) If $x_1, \cdots, x_n$ is a basis of L, then the $C_{n,p}$ elements $f(x_{i_1}, \cdots, x_{i_p})$, $(1 \leqq i_1 < i_2 < \cdots < i_p \leqq n)$ are linearly independent in $\Lambda^p L$.

The following are consequences of (a)–(d):

(e) The $C_{n,p}$ elements of (d) form a basis of $\Lambda^p L$.

(f) If $x_i = x_j$ for some $i \neq j$, then $f(x_1, \cdots, x_n) = 0$.

We now use the following notation. The *exterior product* of $x_1, \cdots, x_p$ is

$$x_1 \wedge \cdots \wedge x_p = f(x_1, \cdots, x_p).$$

Let M be a second linear space over k and A a linear transformation on L into M. This induces a linear transformation, denoted by $\Lambda^p A$, on $\Lambda^p L$ into $\Lambda^p M$ which is determined as follows: If $x_1, \cdots, x_p$ are in L, then

$$(\Lambda^p A)(x_1 \wedge \cdots \wedge x_p) = Ax_1 \wedge \cdots \wedge Ax_p. \tag{1}$$

In terms of matrices, this exterior power $\Lambda^p A$ of A is precisely the pth compound of A.

To see this, we must first introduce some more notation. Let $N=\{1, 2, \cdots, n\}$. We shall denote by H and K subsets of N of exactly p elements and we shall always write $H=\{h_1, h_2, \cdots, h_p\}$ where $h_1<h_2<\cdots<h_p$. Thus there are $C_{n,p}$ possible sets H.

Now let $x_1, \cdots, x_n$ be a basis of L and $y_1, \cdots, y_m$ a basis of M. We must here assume both $0\leqq p\leqq n$ and $0\leqq p\leqq m$. The equations

$$Ax_i = \sum a_{ij}y_j \tag{2}$$

yield the matrix $\|a_{ij}\|$ of A with respect to the given bases x_i, y_j. The $C_{n,p}$ p-vectors $x_H=x_{h_1}\wedge\cdots\wedge x_{h_p}$ form a basis of Λ^pL and the $C_{m,p}$ p-vectors $y_K=y_{k_1}\wedge\cdots\wedge y_{k_p}$ form a basis of Λ^pM. We have

$$\begin{aligned}(\Lambda^pA)x_H &= Ax_{h_1}\wedge\cdots\wedge Ax_{h_p}\\ &= \sum a_{h_1,k_1}\cdots a_{h_p,k_p}y_{k_1}\wedge\cdots\wedge y_{k_p} = \sum a_{H,K}y_K,\\ &\qquad\text{where } a_{H,K}=|a_{h_i,k_j}|,\ i,j=1,\cdots,p.\end{aligned} \tag{3}$$

This shows that the matrix of Λ^pA with respect to the bases x_H and y_K of Λ^pL and Λ^pM is indeed the pth compound of the matrix of A.

It is useful to note that if $A=0$, the zero-transformation on L into M, then $\Lambda^pA=0$, and if $A=I$, the identity transformation on L onto itself, then $\Lambda^pA=I$, the identity on Λ^pL onto itself.

LEMMA. $(\Lambda^pA)(\Lambda^pB)=\Lambda^p(AB)$.

Proof. We have

$$\begin{aligned}(\Lambda^pA)(\Lambda^pB)(x_1\wedge\cdots\wedge x_p) &= (\Lambda^pA)(Bx_1\wedge\cdots\wedge Bx_p) = ABx_1\wedge\cdots\wedge ABx_p\\ &= \Lambda^p(AB)(x_1\wedge\cdots\wedge x_p).\end{aligned}$$

COROLLARY. *If P is a non-singular linear transformation on L onto itself, then Λ^pP is non-singular and $(\Lambda^pP)^{-1}=\Lambda^p(P^{-1})$. If in addition A is a linear transformation on L into itself, then $\Lambda^p(PAP^{-1})$ is similar to Λ^pA. Consequently $|\Lambda^p(PAP^{-1})|=|\Lambda^pA|$.*

Proof. We have $PP^{-1}=I$. By the Lemma, $(\Lambda^pP)(\Lambda^pP^{-1})=\Lambda^pI=I$, hence $(\Lambda^pP)^{-1}=\Lambda^p(P^{-1})$. Likewise, $\Lambda^p(PAP^{-1})=(\Lambda^pP)(\Lambda^pA)(\Lambda^pP)^{-1}$.

It will be recalled that if A is a linear transformation on L into M, then there is induced a linear transformation ${}^tA$, called the transpose of A, on M^* to L^*, where L^* is the dual of L, or space of linear functionals on L to k. When computed with respect to dual bases, the matrix of ${}^tA$ is the usual transpose of the matrix of A. For completeness we mention the following result which is proved in [2, p. 103–4]:

$${}^t(\Lambda^pA) = \Lambda^p({}^tA). \tag{4}$$

This together with the lemma and corollary above yield all the corollaries of [1].

We can now state and prove the Sylvester-Franke theorem.

THEOREM. *If A is a linear transformation on L into itself, then*

$$|\Lambda^p A| = |A|^{C_{n-1,p-1}}. \tag{5}$$

Case 1. The matrix of A is diagonal. Then we have a basis $x_1, \cdots, x_n$ of L such that $Ax_i = a_i x_i$ for scalars a_i. Thus $(\Lambda^p A)x_H = a_H x_H$ where $a_H = a_{i_1} \cdots a_{i_h}$, and so

$$|\Lambda^p A| = \prod_H a_H = (a_1 \cdots a_n)^{C_{n-1,p-1}} = |A|^{C_{n-1,p-1}}.$$

The exponent represents the number of combinations H out of N which contain a specific a_i.

Case 2. The general case. If our field k were the field of complex numbers, we could finish the proof by a continuity argument, using the fact that the matrices with distinct characteristic roots are dense amongst all matrices. But each of these is similar to a diagonal matrix, so the result follows from Case 1 and the Corollary above. An algebraic proof that always works is analogous to this continuity proof. The desired result (4) is an algebraic identity amongst the coefficients of A. Thus it suffices to prove (4) in case $A = \|x_{ij}\|$ is a matrix with independent variables as coefficients. But such a matrix has distinct characteristic roots in an algebraic extension of the field $k(x_{ij})$, hence $PAP^{-1} = B$, a diagonal matrix, and our conclusion follows.

References

1. L. Tornheim, The Sylvester-Franke theorem, this MONTHLY, vol. 59, 1952, pp. 389–91.
2. N. Bourbaki, Algèbre Multilinéaire, Paris, 1948.

CROSS-ASSOCIATIVITY AND ESSENTIAL SIMILARITY†

DAVID ELLIS, The University of Florida

1. Let G be a group under each of the operations $a*b$ and $a\#b$. We consider the two conditions:

(CA) $a\#(b*c) = (a\#b)*c$; for all a, b, c in G.

(ES) There is an element y in G so that for all a, c in G: $a*c = a\#y\#c$.

It is to be noted that we might also consider a dual of (CA) with the cross and star interchanged. As we will show, however, that (CA) and (ES) are equivalent, and since (ES) is symmetric in a and c, it will follow that (CA) and its dual are equivalent.

We denote notions relative to the two operations by subscripts. For example, 1_* is the identity element for $G(*)$ and $(a)_\#^{-1}$ is the inverse of a in $G(\#)$.

† Presented to the Mathematical Association of America; Spring, 1952.

2. THEOREM 1. (CA) *implies* (ES) *where the y in* (ES) *is* $(1_*)_\sharp^{-1}$.

Proof. Write $z = 1_*$ so that $y\#z = 1_\sharp$. Then, by (CA), $a^*c = (a\#1_\sharp)^*c = (a\#y\#z)^*c = (a\#y)\#(z^*c) = (a\#y)\#c$.

THEOREM 2. (ES) *implies* (CA) *and if* (ES) *subsists, the y in* (ES) *is necessarily* $(1_*)_\sharp^{-1}$.

Proof. By direct computation from (ES):

$$(a\#b)^*c = (a\#b)\#(y\#c). \tag{1}$$

$$a\#(b^*c) = a\#(b\#y\#c). \tag{2}$$

Thus, (CA) subsists when (ES) does. But if (CA) subsists, $y = (1_*)_\sharp^{-1}$ by Theorem 1 and cancellation.

THEOREM 3. (ES) *and* (CA) *are equivalent and each implies isomorphism of* $G(*)$ *and* $G(\#)$.

Proof. The first statement combines Theorems 1 and 2. For the remainder, note that if (ES) subsists, then $f(a) = a\#y$ is an isomorphism of $G(*)$ onto $G(\#)$.

3. We have taken $G(*)$ and $G(\#)$ to be groups merely for purposes of familiarity. To obtain the three results, it actually suffices to require only that $G(*)$ and $G(\#)$ are semigroups with identity elements and that the identity of $G(*)$ has an inverse in $G(\#)$. Finally we offer the

CONJECTURE. *If* $G(+, *)$ *and* $G(+, \#)$ *are s-fields, then* (ES) *subsists and, hence,* $G(*)$ *and* $G(\#)$ *are isomorphic.*

CORRECTION

ALBERT WILANSKY, Lehigh University

The following correction should be made in my note, *Two examples in real variables*, this MONTHLY, vol. 60, 1953, p. 317:

For lines 12–15, read: "latter class follows from example 1, the function there constructed being of Baire class I."

Professor Goffman kindly pointed out the inaccuracy of the given text.

CLASSROOM NOTES

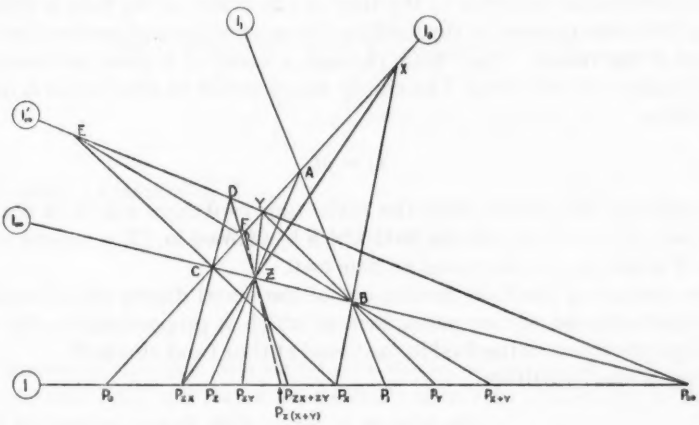
EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

THE DISTRIBUTIVITY THEOREM FOR POINTS ON A LINE

R. C. CLELLAND, Hamilton College

It sometimes seems desirable to prove the distributivity theorem for points on a line without recourse to the theory of quadrangular sets. The following simple proof uses nothing but the usual addition and multiplication constructions together with Desargues' Theorem. The definitions of addition and multiplication of points are those found in chapter six of *Projective Geometry* by Veblen and Young. The notation is that of Figure 1.



THEOREM: *Multiplication is distributive with respect to addition; that is if P_x, P_y, P_z are any three points on the line l for which the operations are defined, we have*

$$P_z \cdot (P_x + P_y) = P_z \cdot P_x + P_z \cdot P_y, \text{ and } (P_x + P_y) \cdot P_z = P_x \cdot P_z + P_y \cdot P_z.$$

Proof: Construction of P_{zx}: Choose arbitrary l_0, l_1, l_∞ on P_0, P_1, P_∞ respectively. Let $l_1 l_0 = A$ and $l_1 l_\infty = B$; $(P_z A) l_\infty = Z$, $(P_x B) l_0 = X$, $(ZX) l = P_{zx}$.

Construction of P_{xy}: $(P_z A) l_\infty = Z$, $(P_y B) l_0 = Y$, $(ZY) l = P_{xy}$.

Construction of P_{zx+xy}: Let $P_\infty Y = l'_\infty$. Let $l_0 l_\infty = C$, $(P_{zx} C) l'_\infty = D$, $(P_{xy} Y) l_\infty = Z$, $(DZ) l = P_{zx+xy}$.

Construction of $P_x + P_y$: $(P_x C) l'_\infty = E$, $(P_y Y) l_\infty = B$, $(EB) l = P_x + P_y$. We do not know that $(EB) l_0 = (ZD) l_0$.

Construction of $P_z(P_x + P_y)$: $(P_z A) l_\infty = Z$, $(P_{x+y} B) l_0 = F$, $(ZF) l = P_z(P_x + P_y)$.

We shall show that l_0, ZD and BE are concurrent in the point F. From this

it follows that $ZD = ZF$, proving left distributivity.

Consider the triangles BZX and EDC. Corresponding sides are seen to meet on l. Hence by the converse of Desargues' Theorem, BE, ZD, and $CX = l_0$ meet in the point F.

Right distributivity may be proved in a similar manner.

FLOW OF FLUID THROUGH A SYSTEM OF VESSELS IN SERIES

S. KULIK, Claremont Men's College

Introduction. The problem arose in connection with some chemical process.

A fluid is treated by filling up a vessel and emptying it at regular intervals. The time it is in the vessel is the time of treatment. In order to accelerate the process it was suggested that the fluid flow continuously through one or several vessels in series, being thoroughly mixed in each vessel.

The distribution function of the time of treatment of the fluid is given, assuming that each element of the fluid has the same chance of leaving the vessel.

Case of one vessel. A fluid flows through a vessel of a given volume v_0 at a constant rate a in unit time. The empty vessel would be filled up in t_0 units of time, where

$$v_0 = at_0 \tag{1}$$

Considering the process from the initial time $t = 0$ there will be in the vessel at a time t, (1) a volume v of the fluid which has flowed in, (2) a volume $v_0 - v$ of the fluid which was in the vessel at time $t = 0$.

The amount of the fluid flowing out of the vessel during the increment of time dt is composed of two parts, each of which is proportional to the corresponding component of the fluid in the vessel at time t and also to dt.

Thus, it can be written

$$d(v_0 - v) = -k(v_0 - v)dt \tag{2}$$

where, as it can be shown, the coefficient $k = a/v_0 = 1/t_0$; the initial condition is $v = 0$ at $t = 0$. Therefore, the solution of the equation is

$$v = v_0(1 - e^{-kt}). \tag{3}$$

The original fluid will theoretically be expelled completely from the vessel only in an infinite period of time, because v reaches the value v_0 only when $t = \infty$. Practically, however, the difference $v_0 - v$ may be considered negligible at some finite value of t.

When the initial fluid is expelled the process is considered as steady, and the proportion of each unit of the volume which had been treated for not less than time t is, according to equation (3),

$$F(t) = 1 - e^{-kt} \tag{4}$$

and the proportion which received treatment between times t and $t+dt$ is:

$$dF(t) = ke^{-kt}dt, \tag{5}$$

that is $f(t) = ke^{-kt}$ is the frequency function of the time of treatment of the fluid flowing continuously through a vessel.

Case of any number of vessels in series. It is assumed that the process is steady. First consider two vessels of the same size in series. The fraction of a unit volume which is treated in the first vessel for the period of time t to $t+dt$ is represented by $dF_1(t) = f_1(t)dt$ and the fraction which also undergoes treatment for the period time t_1 to t_1+dt_1 in the second vessel is $dF_1(t)dF_1(t_1) = f_1(t)f_1(t_1)dtdt_1$, where $dF_1(x) = dF(x)$ and $f_1(x) = f(x)$.

Then, by the usual technique for finding the distribution of a sum we find that the fraction of the unit volume which has undergone treatment in the two vessels for the total period of time T to $T+dT$ will be represented by:

$$\begin{aligned} dF_2(T) &= f_2(T)dT = dT\int_0^T f_1(x)f_1(T-x)dx \\ &= k^2dT\int_0^T e^{-kx}e^{-k(T-x)}dx = k^2Te^{-kT}dT. \end{aligned} \tag{6}$$

Hence, writing t instead of T

$$dF_2(t) = k^2te^{-kt}dt = f_2(t)dt, \tag{7}$$

where

$$f_2(t) = k^2te^{-kt}, \qquad 0 \leqq t \leqq \infty, \tag{8}$$

is the frequency function. Repeating now the same arguments with the functions $F_1(t)$ and $F_2(t)$, and then more generally with $F_1(t)$ and $F_{n-1}(t)$ we shall arrive at the general result

$$dF_n(t) = \frac{k^n}{(n-1)!}t^{n-1}e^{-kt}dt \tag{9}$$

and

$$f_n(t) = \frac{k^n}{(n-1)!}t^{n-1}e^{-kt}, \qquad 0 \leqq t \leqq \infty, \tag{10}$$

where $f_n(t)$ is the frequency function for the time of treatment of the fluid flowing through n vessels in series.

The average time of treatment is

$$\mu_{1,n} = \int_0^\infty tf_n(t)dt = n/k = nt_0, \tag{11}$$

the variance

$$V_n = \int_0^\infty (t - nt_0) f_n(t)\,dt = nt_0^2 = n/k^2 \tag{12}$$

and the standard deviation

$$\sigma_n = t_0\sqrt{n}. \tag{13}$$

Thus when n vessels in series are used the average time of treatment equals the time required to fill up with fluid all the empty vessels at a given rate of flow, and the standard deviation is $\sqrt{n}$ of this time.

Hence, it can be deduced that if n vessels have a total constant volume equal to that in the case of one vessel, the formulas (10)–(13) must be rewritten as follows:

$$f_n(t) = \frac{(nk)^n}{(n-1)!}\, t^{n-1} e^{-nkt} \tag{14}$$

$$\mu_{1,n} = 1/k = t_0 \tag{15}$$

$$V_n = 1/nk^2 = t_0^2 n \tag{16}$$

$$\sigma_n = 1/k\sqrt{n} = t_0/\sqrt{n}. \tag{17}$$

Thus the average remains constant while the variance is one nth of that for one vessel.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1081. *Proposed by A. E. Livingston, Institute for Advanced Study, Princeton, N. J.*

Let $f(x_1, \cdots, x_n)$ be the nth order determinant $|a_{ij}|$ with $a_{ii}=x_i$ and $a_{ij}=1$ for $i\neq j$. Clearly, $f(x_1, \cdots, x_n)$ is symmetric in $x_1, \cdots, x_n$ Find its representation in terms of the elementary symmetric functions in $x_1, \cdots, x_n$.

E 1082. *Proposed by L. L. Pennisi, University of Illinois*

Show that $\lim_{n\to\infty} n(\sqrt{n}/n!)^{1/n} = e$.

E 1083. *Proposed by F. J. Duarte, Caracas, Venezuela*

Suppose that $x^n + y^n = z^n$, where n is a positive integer and x, y, z are positive real numbers. Show that $(xy/z^2)^n < 2/5$.

E 1084. *Proposed by R. E. Shafer, University of California*

Trygve Nagell in his book *Elementary Number Theory* shows that

$$\sum_{p \leq x} (1/p) > \log\log x - 1,$$

where the summation is over all primes p not exceeding x. Establish the sharper inequality

$$\sum_{p \leq x} (1/p) > \log\log x + 1 - \pi^2/6.$$

E 1085. *Proposed by Josef Langr, Prague, Czechoslovakia*

The perpendicular bisectors of the sides of a quadrilateral Q form a quadrilateral Q_1, and the perpendicular bisectors of the sides of Q_1 form a quadrilateral Q_2. Show that Q_2 is similar to Q and find the ratio of similitude.

SOLUTIONS

A Transcendental Number

E 1050 [1953, 40]. *Proposed by S. H. Gould, Purdue University*

If $c_0 = 1$, $c_{i+1} = 2^{c_i}$, prove that $\sum_{i=1}^{\infty} c_i^{-1}$ is transcendental.

Solution by J. V. Whittaker, U. C. L. A. If $k \geq 1$, then $k \leq c_{k-1}$ and $kc_{k-1} \leq c_{k-1}^2 \leq 2^{c_{k-1}} = c_k$. Hence, $2^{kc_{k-1}} \leq 2^{c_k}$, or $c_k^k \leq c_{k+1}$. Suppose $\xi = \sum_{i=1}^{\infty} 1/c_i$ were algebraic of degree $n > 1$, and let $\xi_k = \sum_{i=1}^{k} 1/c_i = p_k/q_k$, where p_k and $q_k = c_k$ are integers. Then $\xi - p_k/q_k = \sum_{i=k+1}^{\infty} 1/c_i < 2/c_{k+1} \leq 2/c_k^k$, and $\lim_{k\to\infty} q_k^n(\xi - p_k/q_k) = \lim_{k\to\infty} 2q_k^n/q_k^k = 0$. But this contradicts the hypothesis that ξ was algebraic of degree $n > 1$, by Liouville's theorem. Now suppose $\xi = p/q$, p and q integers, and let $c_k > q$. Then the integer $\xi q c_k - \xi_k q c_k = q c_k \sum_{i=k+1}^{\infty} 1/c_i < 2qc_k/c_{k+1} \leq 2q/c_k^{k-1} < 2/c_k^{k-2} < 1$ if $k \geq 3$, which is a contradiction. Therefore, ξ is transcendental.

Also solved by W. E. Briggs, Vern Hoggatt, John Jones, Jr., M. S. Klamkin, A. E. Livingston, Albert Wilansky, and the proposer.

N Objects in *B* Boxes

E 1051 [1953, 114]. *Proposed by S. W. Golomb, Harvard University*

Given N objects and B boxes, what is a necessary and sufficient condition for at least two boxes to contain the same number of objects?

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If no two of the boxes contain the same number of objects, then the total number of objects is

$$0 + 1 + 2 + \cdots + (B - 1) = B(B - 1)/2$$

or any greater integer. Therefore a necessary and sufficient condition for at least two boxes to contain the same number of objects is that $N < B(B-1)/2$.

Also solved by N. J. Fine, L. A. Fulk, H. M. Gehman, Leo Moser, Azriel Rosenfeld, I. R. Savage, and the proposer.

Three Related Loci

E 1052 [1953, 114]. *Proposed by H. H. Berry, U. S. Army Engineering Corps*

Let AOB be a fixed diameter of a given circle (O), and let P be any point on the circle. Denote by Q the foot of the perpendicular from P on AB and by R the foot of the perpendicular from O on PA. Let PQ and RO intersect in N, and let QR and PO intersect NA in L and M, respectively. Find the loci of points L, M, and N as P moves along the given circle.

Solution by D. C. B. Marsh, University of Colorado. Take O as origin and OB as positive x-axis. If r is the radius of the given circle and (u, v) the coordinates of point P, we readily find the coordinates of M, N, L to be

$$M: \quad (-u^2/r, \ -uv/r),$$

$$N: \quad (u, \ -uv/(r - u)),$$

$$L: \quad (u(2r - u)/(r - 2u), \ -uv/(r - 2u)).$$

Eliminating u and v from each of these we get:

$$\text{locus of } M: \quad x^2 + y^2 + rx = 0,$$

$$\text{locus of } N: \quad y^2(r - x) = x^2(r + x),$$

$$\text{locus of } L: \quad [(x + r)^2 + y^2](x^2 - 3y^2) = r^2y^2.$$

Also solved by W. B. Carver, Charlotte Froese, J. D. Haggard, J. R. Hatcher, Douglas Holdridge, A. E. Livingston, B. Martin, L. V. Mead, C. S. Ogilvy, M. Perisastri, A. Sisk, O. E. Stanaitis, and the proposer.

Editorial Note. The locus of M is a circle on AO as diameter. The locus of N is a right strophoid with the end of its loop at A, its node at O, and its asymptote along the tangent at B to the given circle. The locus of L is a unicursal quartic symmetrical in the x-axis and having the lines $x^2 - 3y^2 = 0$ as asymptotes; it has a crunode at O and a tacnode at A.

Evaluation of a Certain Symmetrical Determinant

E 1053 [1953, 115]. *Proposed by H. S. Shapiro, Chatham, N.J.*

Evaluate the determinant

$$\begin{vmatrix} x & x-1 & \cdots & x-k \\ x-1 & x & \cdots & x-k+1 \\ \cdot & \cdot & \cdots & \cdot \\ x-k & x-k+1 & \cdots & x \end{vmatrix}.$$

Solution by N. J. Fine, University of Pennsylvania. Subtract the second row from the first, the third from the second, $\cdots$, the $(k+1)$th from the kth. Now add the first column to the second, third, $\cdots$, $(k+1)$th to get a triangular determinant with diagonal elements 1, 2, 2, $\cdots$, 2, $2x-k$. Hence the value is $2^{k-1}(2x-k)$.

Also solved by Norman Anning, A. P. Bakshi, C. R. Berndtson, R. L. Blair, A. P. Boblétt, L. F. Boron, Julian Braun, F. J. Duarte, Charlotte Froese, J. D. Haggard, J. R. Hatcher, J. Heading, Vern Hoggatt, R. Huck, A. R. Hyde, H. I. James, P. Q. Jennett, John Jones, Jr., J. H. Kaplan, C. W. Karns, M. S. Klamkin, A. E. Livingston, D. C. B. Marsh, Leo Moser, C. S. Ogilvy, Hyman Orlin, S. Parameswaran, F. D. Parker, M. Perisastri, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, Edward Saibel, Nathan Schwid, O. E. Stanaitis, D. D. Strebe, C. F. Strobel, D. R. Sudborough, J. A. Tierney, C. W. Trigg, J. V. Whittaker, F. H. Young, and the proposer.

Probability of Imaginary Roots of a Quadratic Equation

E 1054 [1953, 115]. *Proposed by F. D. Parker, St. Lawrence University*

Find the probability that the roots of the quadratic equation

$$x^2 + bx + c = 0, \qquad -k < b,c < k,$$

are imaginary.

Solution by H. M. Gehman, University of Buffalo. We shall generalize the problem slightly by imposing the conditions: $-s<b<s$, $-t<c<t$. The required probability is the fraction of area in the (b,c)-plane within the rectangle whose vertices are $(\pm s, \pm t)$ which lies above the parabola $4c=b^2$. If $s^2 \leqq 4t$, this probability is $p=(12t-s^2)/24t$. If $s^2 \geqq 4t$, $p=2\sqrt{t}/3s$. Now if $s=t=k$, we have: if $k \leqq 4$, $p=(12-k)/24$, and if $k \geqq 4$, $p=2/3\sqrt{k}$.

Also solved by Leon Bankoff, Julian Braun, H. K. Crowder, Fred Discepoli, A. L. Epstein, N. J. Fine, A. D. Fleshler, A. R. Hyde, A. E. Livingston, Leo Moser, L. A. Ringenberg, William Small, W. R. Smythe, Jr., F. H. Young, and the proposer.

Moser called attention to H. Levy and L. Roth, *Elements of Probability*, pp. 84–85. Discepoli noted the interesting paradox (gotten by taking k infinitely large) that the probability that the roots of the unrestricted real quadratic equation be imaginary is zero, or the probability that the roots be real is certainty.

Quadrangles with Incircles

E 1055 [1953, 115]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The center of the four circles passing through triples of vertices of a quadrangle $ABCD$ inscribed in (circumscribed about) a circle (O) are the vertices of a quadrangle $A'B'C'D'$ inscribed in (circumscribed about) a circle (O').

Editorial Note. The part of the theorem not in parentheses is trivially true, since in this case the quadrangle $A'B'C'D'$ reduces to the point O.

To prove the part of the theorem in parentheses let AB and CD intersect in E and let (O'') be the circle inscribed in triangle BCE. Let CB'', parallel to DB, intersect BE in B'', and let BC'', parallel to AC, intersect CE in C''. Then $B''C''$ and DA are corresponding lines under the negative homothety set up by circles (O'') and (O). It follows that $B''C''$ is tangent to (O''), or that $CC''B''B$ possesses an incircle. Now one can easily show that $A'B'C'D'$ is similar to $CC''B''B$, since corresponding sides and diagonals are perpendicular so as to make corresponding angles equal. Therefore $A'B'C'D'$ possesses an incircle.

Also solved, trigonometrically, by Josef Langr.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4553. *Proposed by D. J. Newman, Harvard University*

Consider any sequence $\{a_n\}$ of real numbers. Prove

$$\sum_{n=1}^{\infty} a_n \leqq \sqrt{2} \sum_{n=1}^{\infty} \sqrt{\frac{a_n^2 + a_{n+1}^2 + \cdots}{n}}.$$

4554. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Find the general solution of the differential equation

$$\left(\frac{dy}{dx} + Py\right)^P = R\left(\frac{dy}{dx} + Qy\right)^Q,$$

where P, Q and R are constants.

4555. *Proposed by R. M. Redheffer, University of California at Los Angeles*

If $\tanh u = \tanh^2(\pi\sqrt{z}\sin\theta)$ and $\tanh v = \tan^2(\pi\sqrt{z}\cos\theta)$, then the function

$$F(z) = \int_{-\pi}^{\pi} \coth(u+v)d\theta - 2\cot \pi z$$

has simple poles at the (positive and negative) composite integers and is regular everywhere else.

4556. *Proposed by H. D. Grossman, New York City*

Suppose an election results in km votes for A and kn votes for B. In how many orders may votes be cast so that A's vote is always at least m/n times B's? Prove the following formula (cited in *Scripta Math.*, v. 16, pp. 207–212, and in *Math. Reviews*, October 1951, p. 665)

$$p_k = \sum \frac{F_1^{k_1}F_2^{k_2}\cdots}{k_1!k_2!},$$

in which $k_1+2k_2+\cdots=k$ and $F_j = \binom{jm+jn}{jm}/j(m+n)$, and the summation is taken over all partitions of k.

4557. *Proposed by D. H. Browne, Buffalo, N. Y.*

Let $f(n)$ be the number of ways the integer n can be decomposed into dissimilar factors [e.g. $f(1)=1$, $f(45)=3$ because $1\cdot 45 = 3\cdot 15 = 5\cdot 9$ but not $3\cdot 3\cdot 5$]. Evaluate

$$\sum_{n=0}^{\infty} \frac{f(2n+1)}{(2n+1)^2}.$$

SOLUTIONS

Basis in a Banach Space

4478 [1952, 186]. *Proposed by Albert Wilansky, Lehigh University*

A basis in a Banach space B is a set of elements $\{x_n\}$ such that every element in the space is a unique, finite or infinite, linear combination of elements of the set.

(1) Prove that the x_n are isolated.

(2) If $\sum x_n$ is an element of B, show that there is a divergent sequence $\{s_n\}$ of numbers such that $\sum s_n x_n$ is an element of B.

Correction by G. G. Lorentz. The solution of (1) as given in the May issue [1953, 338] is incomplete and should be replaced by the following: Let

$$x = \sum_{i=1}^{\infty} a_i x_i$$

be the representation of an element x of the space by means of the basis. Then $a_i = a_i(x)$ is a bounded linear functional (Banach, *Théorie des operations linéaires*, Warsaw 1932, p. 111), which vanishes for $x = x_j$, $j \neq i$, and is equal to 1 for $x = x_i$. Hence (*loc. cit.*, p. 58) x_i is at a positive distance from the linear set spanned by the x_j, $j \neq i$, and is therefore isolated from these x_j.

Linear Combination of Vectors with Non-negative Coefficients

4490 [1952, 332]. *Proposed by L. M. Blumenthal, University of Missouri*

Let S be a set of more than $2n$ pairwise distinct (non-null) vectors of Euclidean n-space E_n. If each vector of E_n is expressible as a linear combination of vectors of S with non-negative coefficients, then S contains a subset R of at most $2n-1$ vectors such that R has this same property. (This is an extension of no. 4395 [1952, 46].)

Solution by V. L. Klee, Jr., The Institute for Advanced Study. For a subset X of E_n, the following assertions are equivalent: (a) each vector of E_n is expressible as a linear combination of vectors of X with non-negative coefficients: (b) the origin is interior to the convex hull of X. Now with S as given, it is clear that some finite subset X of S has property (a). But then, since (a) is equivalent to (b), the desired conclusion is implied by the following result of C. V. Robinson (Spherical theorems of Helly type and congruence indices of spherical caps, *Amer. Jour. Math.*, vol. 64 (1942), p. 263): *If q is interior to a convex polyhedron P of E_n with vertices $p_1, p_2, \cdots, p_m (m \geqq 2n-1)$ then either* (1) *q is interior to a sub-polyhedron of P with $2n-1$ or fewer vertices, or* (2) *P has exactly $2n$ vertices collinear in pairs with q.*

Also solved by J. W. Gaddum and the Proposer.

The Brocard Angle

4491 [1952, 333]. *Proposed by C. S. Venkataraman, Tricur, South India*

If ω denotes the Brocard angle of a triangle ABC, prove that

(i) the sides are equal when $\cot \omega = \sqrt{3}$.

(ii) the squares of the lengths of the sides, a^2, b^2, c^2, are in arithmetic progression when $\cot \omega = 3 \cot B$.

Solution by W. B. Carver, Cornell University. It is well known (see Johnson, *Modern Geometry*, p. 266) that

$$\cot \omega = \cot A + \cot B + \cot C,$$
$$\cot A = (b^2 + c^2 - a^2)/4\Delta,$$

where Δ is the area of the triangle ABC, with analogous formulas for $\cot B$ and $\cot C$. Therefore $\cot \omega = (a^2+b^2+c^2)/4\Delta$.

(i) When $\cot \omega = \sqrt{3}$, we have, after expressing Δ in terms of the sides and simplifying,

$$(b^2 - c^2)^2 + (c^2 - a^2)^2 + (a^2 - b^2)^2 = 0.$$

Since a, b, c are real and positive, this implies $a = b = c$.

(ii) From $\cot \omega = 3 \cot B$, we obtain

$$2b^2 = a^2 + c^2,$$

which makes b^2 the arithmetic mean between a^2 and c^2.

Both conditions are necessary and sufficient.

Also solved by F. A. Alfieri, Louisa S. Grinstein, L. M. Kelly, M. S. Klamkin, J. D. E. Konhauser, Josef Langr, Margaret Olmsted, M. Perisastri, L. A. Ringenberg, F. Underwood, Chih-yi Wang, Alan Wayne, and the Proposer.

A Locally Increasing Function

4492 [1952, 333]. *Proposed by G. Lumer, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

A real function $f(x)$ is said to be locally increasing at the point x_0 if there exists a neighborhood U of x_0 such that for $x \in U$,

$$\frac{f(x) - f(x_0)}{x - x_0} \geqq 0.$$

Show that there exist functions $f(x)$, defined and continuous on a closed interval $[a, b]$, locally increasing on a set D, everywhere dense on $[a, b]$, but such that $f(b) < f(a)$.

Solution by E. M. Beesley, University of Nevada. For $0 \leqq x \leqq 1$, let $F(x) = -C(x)$ where C is the well known Cantor function. (See Titchmarsh, *Theory of Functions*, p. 366; or Hobson, *the Theory of Functions of a Real Variable*, v. I, p. 368.) This function is continuous and has a derivative almost everywhere. Then $F(1) = -1 < 0 = F(0)$. If x is not in Cantor's set, $F'(x) = -C'(x) = 0$. Since Cantor's set is nowhere dense, F is a solution to the proposed problem.

If we let $F(x) = -C(x) + ax$, where $0 < a < 1$, we have a better result in the sense that $F'(x) > 0$ on the same everywhere dense set.

Also solved by Henry Furstenberg, L. M. Kelly, M. S. Klamkin, J. H. Michael, C. L. Perry and W. S. Snyder, George Piranian, Edgar Reich, L. A. Ringenberg, Arthur Rosenthal, W. R. Scott, and the Proposer.

An Irrational Number

4493 [1952, 412]. *Proposed by Paul Erdös, American University, Washington, D. C.*

For each integer n let $\sigma(n)$ be the sum of its divisors. Prove that

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!}$$

is irrational.

Solution by J. B. Kelly, Michigan State College. Suppose that $h = \sum_{n=1}^{\infty} \sigma(n)/n! = r/s$, where r and s are positive integers and $(r, s) = 1$. Let p be any prime larger than s and larger than 6. Then

$$h = \sum_{n=1}^{p-1} \frac{\sigma(n)}{n!} + \sum_{n=p}^{\infty} \frac{\sigma(n)}{n!},$$

whence

$$(p-1)!h = (p-1)! \sum_{n=1}^{p-1} \frac{\sigma(n)}{n!} + \sum_{c=0}^{\infty} \frac{\sigma(p+c)}{p(p+1)\cdots(p+c)}.$$

Now $(p-1)!h$ and $(p-1)! \sum_{n=1}^{p-1} \sigma(n)/n!$ are integers. We shall obtain a contradiction by showing that

$$k = \sum_{c=0}^{\infty} \frac{\sigma(p+c)}{p(p+1)\cdots(p+c)} \tag{1}$$

cannot be an integer.

We have

$$\sigma(p)/p = 1 + 1/p, \tag{2}$$

while $\sigma(p+c) < 1+2+3+\cdots+(p+c) = \frac{1}{2}(p+c)(p+c+1)$, so that

$$\begin{aligned} \sum_{c=1}^{\infty} \frac{\sigma(p+c)}{p(p+1)\cdots(p+c)} &< \sum_{c=1}^{\infty} \frac{p+c+1}{2p(p+1)\cdots(p+c-1)} \\ &< \sum_{c=1}^{\infty} \frac{p+2}{2p^c} < \frac{p+2}{2(p-1)}. \end{aligned} \tag{3}$$

Also for $p > 6$, $(p+2)/2(p-1) < (p-1)/p$. Finally we have from (1), (2) and (3),

$$1 < k < 1 + \frac{1}{p} + \frac{p+2}{2(p-1)} < 2.$$

Also solved by Robert Breusch, Leonard Carlitz, N. J. Fine, Fritz Herzog, Leo Moser, D. J. Newman, A. Oppenheim, and L. L. Pennisi.

Editorial Note. Several contributors address themselves to other problems of the form $h = \sum a_n/n!$. Moser proves that h is irrational for $a_n = \phi(n)$. Oppenheim proves h irrational for $a_n = \phi(n)\sigma(n)$. See also the Proposer's problem 4518, [1953, 47]. The Proposer is also interested in determining whether or not $\sum \sigma(n)/2^n$ is rational.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

An Introduction to Mathematical Thought. By E. R. Stabler. Cambridge, Mass., Addison-Wesley Publishing Co., 1953. 20+268 pages. $4.50.

Our schools and colleges offer a multitude of subjects which presuppose, to a greater or lesser degree, the ability to think logically. It seems not to have occurred to many of those who determine curricula that pupils are not born with this ability, relatively few acquire it incidentally, and not a few of them would acquire it rather easily if it were only taught them. It is a common occurrence for students in a freshman mathematics course to be not only unprepared in algebra and trigonometry, but to be utterly ignorant of the essentials of logical reasoning, and this is an even greater handicap in a course in analysis. What is more important, the failure of our institutions to provide some training for *all* in the rudiments of logic is responsible, to some extent, for the ubiquitous written and spoken rubbish which passes for sound reasoning, even among "educated" people and in scientific literature. This state of affairs is, no doubt, highly profitable for some individuals who benefit from the ignorance of others, and of no concern to those who should know better but who find it to their advantage to condone mental laziness. But those who believe that the rudiments of logic and scientific method should be as much a part of the educated man, especially in this scientific age, as, *e.g.*, the ability to read, write, and add, would do well to advocate a required course for freshmen, based on a book such as that under review. If high school students, including those not planning to attend college, took a similar course on a more elementary level, it would equip them for the "outside world" better than some of the "practical" subjects they are taught now.

Stabler's "introductory and somewhat exploratory" book consists of two parts. The first is called "Mathematical thought in relation to logic and science," and the following is a brief indication of the topics appearing in the various chapters: 1. The logical dependence of mathematical results on postulates and definitions, with illustrations from non-Euclidean geometries and modulo n arithmetic. 2. Criticism of Euclid from the standpoint of modern rigor; the value of postulational organization in various branches of mathematics, in science, and in more general realms. 3 and 4. The essentials of logical reasoning from the standpoint of symbolic logic; errors in reasoning, with illustrations from contemporary life. 5. Application of logical reasoning to the formulation of scientific theories; the scientific method. The second part, "Some significant postulational systems, concepts, and methods," contains the following chapters: 6. Development of a part of elementary algebra as a postulational system. 7. A finite geometry as an illustration of an abstract geometrical system. 8. Discus-

sion of consistency, independence, completeness, and categoricalness of postulates, with illustrations, including various field postulates. 9. Groups, rings, and Boolean algebras. 10. General and equivalence relations, linearly and partially ordered sets, and lattices. 11. A survey of the foundations of mathematics, with brief sketches of the various schools. The exercises are interesting and well chosen, and the text is alive.

There are very few misprints, and only the one on the bottom of page 96, where "$x \in A$" should be replaced by "$x \in A$," might cause the student a little trouble. On the bottom of page 143, it is stated that "the lines of projective geometry are thought of as made up continuously of points"; the existence of finite projective geometries shows, however, that this is false. On page 183, the author should have been more specific as to the applications of the theory of finite groups in the theory of algebraic equations. Bolyai's contribution to non-Euclidean geometry is somewhat slighted, the absence of biographical notice in connection with the names Abel and De Morgan is conspicuous, and bouquets are thrown to mathematicians prominent in America. These, however, are minor imperfections in the book of many merits.

F. BAGEMIHL
Rochester, N. Y.

Advanced Mathematics in Physics and Engineering. By Arthur Bronwell. McGraw-Hill Book Company, 1953. xvi+475 pages. $6.00.

This text by an electrical engineer is intended for senior or graduate use in the fields mentioned. Chapter headings are as follows: 1. Infinite Series. 2. Complex Numbers and Hyperbolic Functions. 3. Fourier Series and Fourier Integral. 4. Ordinary Differential Equations. 5. Series Solution of Differential Equations —Bessel and Legendre Equations. 6. Partial Differentiation. 7. Elastic Vibrations and Electric Oscillations—Systems with Lumped Elements. 8. Vibrations in Systems with Distributed Elements. 9. Lagrange's Equations. 10. Vector Analysis. 11. Solutions of the Wave Equation. 12. Heat Flow. 13. Dynamics of Fluids. 14. Electromagnetic Theory. 15. Functions of a Complex Variable. 16. Complex Roots of Polynomials and Dynamic Stability. 17. Laplace Transformations. Chapters 2 and 6 are short, chapters 5, 15 and 17 are long. Each chapter ends with a bibliography, and, except 16, with an adequate set of problems, some with answers. Illustrative examples however are few and untypically easy.

The major comment called for by this text concerns the generally unsatisfactory quality of the exposition. There is much carelessness, linguistic and mathematical, mostly in the first half of the book. Many statements are wrong, still more are meaningless from lack of stated assumptions, and quite often a sentence means just the opposite of what the author intended. Chapter 11, signalled out in the preface as an innovation, attempts too many things simultaneously and is merely confusing. Errors range all the way from confusion of necessary with sufficient conditions to the implication that drums have rec-

tangular membranes.

The last chapters are less careless than the rest, and the reviewer profited from the discussions of wave guides and of dynamic stability. There is about the whole a commendable balance as between fields treated; and the attempt to do justice and bring unity to such diverse problems in less than 500 uncrowded pages is laudable. However this reviewer would not recommend this book as a whole to any beginner in this area; some of the later sections should have value for the experienced and careful reader.

E. H. CUTLER
Lehigh University

NEW BOOKS RECEIVED

Logic and Language. By A. G. N. Flew. New York, Philosophical Library, 1953. 8+242 pages. $4.75.

Elements of Algebra. By Vaughn B. Caris. Boston, Ginn and Company, 1953. 5+307 pages. $3.30.

An Introduction to Statistical Science in Agriculture. By D. J. Finney. New York, John Wiley and Sons, 1953. 179 pages. $3.75.

The Teaching of Secondary Mathematics. By Claude H. Brown. New York, Harper and Brothers, 1953, 11+388 pages. $4.00.

Cycles of Recurring Decimals. By D. R. Kaprekar. Groningen (Netherlands), P. Noordhoff Ltd.

Beginning Algebra for College Students, Second Edition. By Lloyd L. Lowenstein. New York, John Wiley and Sons Co., 1953. 13+279 pages. $3.50.

Sampling Techniques. By William G. Cochran. New York, John Wiley and Sons Co., 1953. 14+330 pages.

Plane Trigonometry with Tables, Second Edition. By Donald H. Ballou and Frederick H. Steen. Boston, Ginn and Company, 1953. 5+150+10+84 pages. $3.25.

Plane and Spherical Trigonometry with Tables, Second Edition. By Donald H. Ballou and Frederick H. Steen. Boston, Ginn and Company, 1953. 6+205 +13+84 pages. $3.50.

Introduction to Logic. By Irving M. Copi. New York, The Macmillan Company, 1953. 16+472 pages. $4.00.

Calculus and Analytic Geometry, Second Edition. By George B. Thomas. Cambridge, Mass., Addison-Wesley Publishing Company, 1953. 11+731 pages $8.50.

Einführung in die höhere Mathematik. By Feigl und Rohrback. Berlin, 1953. Ladenpreis: Ganzleinen DM 26.80.

Introduction to the Theory of Statistics. By Victor Goedicke. New York, Harper and Brothers, 1953. 12+286 pages. $4.50.

College Algebra, Fourth Edition. By William L. Hart. Boston, D. C. Heath and Company, 1953. 10+480 pages. $3.50.

Man and His Physical Universe. By Richard Wister. New York, John Wiley and Sons, Inc., 1953. 15+488 pages. $4.75.

Mathematical Recreations, Second Revised Edition. By Maurice Kraitchik. New York, Dover Publications, Inc., 1953. $1.60 paperbound, $3.00 clothbound.

Cardano the Gambling Scholar. By Oystein Ore. Princeton, N. J., Princeton University Press, 1953. 14+259 pages. $4.00.

Infinity. By Lillian R. Lieber. New York, Rinehart and Company, Inc., 1953. 359 pages. $5.00.

Real Functions. By Casper Goffman. New York, Rinehart and Company, Inc., 1953. 12+263 pages. $6.00.

Fluid Dynamics, Vol. IV. By M. H. Martin. New York, McGraw-Hill Book Company, 1953. 5+186 pages. $7.00.

Rinehart Mathematical Tables, Formulas and Curves, Enlarged Edition. By Harold D. Larsen. New York, Rinehart and Company, Inc., 1953. 8+280 pages. $2.00.

Theory of Functions of Real Variables. By Henry P. Thielman. New York, Prentice-Hall, Inc., 1953. 11+209 pages. $5.00.

Table of the Arctan x, Second Edition (Applied Mathematics Series 26). By Bureau of Standards. Washington, Government Printing Office. 171 pages, buckram bound, $1.75.

Principles of Mathematical Analysis, International Series in Pure and Applied Mathematics. By Walter Rudin. New York, McGraw-Hill Book Company. ix+227 pages, 1953. $5.00.

College Algebra. By R. H. Bardell and Abraham Spitzbart. Cambridge, Massachusetts, Addison-Wesley Publishing Company, 1953. 9+197 pages. $3.50.

Fundamentals of College Mathematics. By Johnson, McCoy and O'Neill. New York, Rinehart and Company, Inc., 1953, 14+479 pages. $6.00.

Readings in Philosophy of Science. By Philip P. Wiener. New York, Charles Scribners Sons, 1953. 9+645 pages. $5.50.

Wiskundige Tafels. By P. Wijdenes. Groningen, Holland, P. Noordhoff. 8+279 pages. fl. 5,25.

Tables for Rocket and Comet Orbits (Bureau of Standards Applied Mathematics Series 20). By Samuel Herrick. Washington, D. C., Government Printing Office. 100 pages. $.75.

Calculus. By C. R. Wylie, Jr. New York, McGraw-Hill Book Company, 1953. 10+565 pages. $6.00.

First Course in Abstract Algebra. By Richard E. Johnson. New York, Prentice-Hall, Inc., 1953. 8+257 pages. $5.50.

Flow Functions and Their Application to Some Problems of Conformal Mapping. By H. Bolder. Holland, The University of Leiden.

Foundations of Engineering, Part II. By J. Alfred Benner and K. D. Larsen. Mimeographed Form, Copyright 1952. 500 pages.

Equazioni Differenziali, Second Edition. By Francesco G. Tricomi. Turine,

Italy, Einaudi Publishers, 1953. 353 pages. L. 4000.

A Survey of Modern Algebra, Revised Edition. By Garrett Birkhoff and Saunders MacLane. New York, The Macmillan Company, 1953. 11+472 pages. $6.50.

Trigonometry. By John Randolph. New York, The Macmillan Company. 10+220 pages, 1953. $3.00.

Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory, Applied Mathematics Series 19. By the Bureau of Standards. Washington, D. C. Government Printing Office, May, 1952. 11+427 pages. $3.25.

Studies in Econometric Method, No. 14 in the Cowles Commission Monograph Series. By William C. Hood and Tjalling C. Koopmans. New York, John Wiley and Sons, Inc., June, 1953. 323 pages. $5.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

THE ELIMINATION OF THE HIGH COST OF COMPTOMETRY

E. I. GALE, Norland, Ontario

"I took a few pieces of string and an old tin can and made myself a Ford car and the darn thing ran."—The late Dr. Jas. S. Gale of Korea.

His nephew has just assembled the same primitive materials to evolve a remarkably simple and efficient method of carrying out multiplications of astronomical magnitudes. The scheme in a nutshell is to combine the principle of Genaille's rods (see this MONTHLY, February, 1953) and the suanpan (see your Chinese laundry man). The former invention yields the partial products, while the abacus is well adapted for assembling them into the final product.

An example in addition may be helpful to those readers unfamiliar with the oriental method of using the abacus for addition. The bar running athwart the rods carrying the beads separates "heaven" from "hell." On the "heaven" side sliding on each rod are a pair of "quints" while on each rod in "hell" are five units. The rods in order from the right follow the Arabic system of counting units, tens, hundreds, *etc.* To "clear" the board, push quints and units as far from the central bar as possible. The board now reads zero. Let us add 269. Bring a quint down and four units up on the units rod for the 9. On the tens rod bring down a quint and raise a unit for the 6. Finally on the hundreds rod raise two units for the 2. The suanpan reads 269. We now will add 584. Obviously we can't raise four units on the units rod so we depress the other quint and lower a

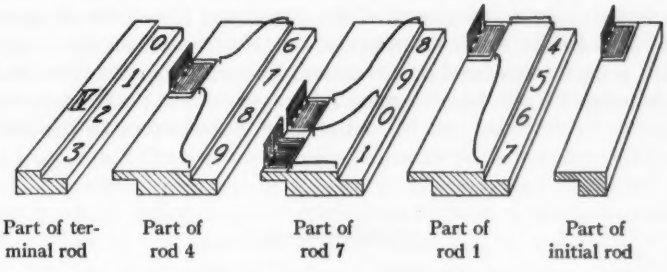
unit. Now a universal abacus rule forbids two quints to rest at the central bar so we raise these two quints and raise a compensating unit on the tens rod. The board now reads 273. To add 9 tens the same method is followed, *viz.*, lower a quint on the tens rod and raise three units. However by the rule mentioned above the two tens quints are pushed to the top and a unit raised on the hundreds rod. The board now reads 353. The reader will observe here that it would be more economical with respect to beads to replace the five units on the tens rod by pulling down a quint on the same rod. This we do. This is a second rule and is invariably followed. The state of the board will now be 3 units on the hundreds rod, 1 quint on the tens rod, and 3 units on the units rod. Finally depress a quint on the hundreds rod yielding the desired 853. In practice this operation is performed much more quickly than the description would lead one to suppose.

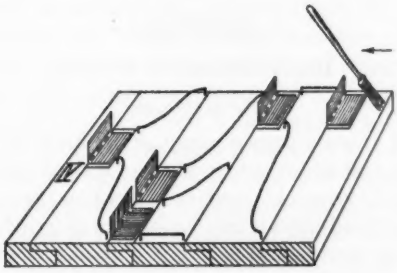
The abacus is kept in the horizontal plane while being manipulated as is also the author's modification of Genaille's rods, which will be explained here. To follow the description use the diagram on page 140 of the February number of this MONTHLY. Genaille's arrows indicate the significant figures which are used in the computation (note an error in the "one" rod, row 8: 8 and 9 should point to 6, while 0 to 5 should point to 7).

In the author's instrument, shields cover all figures which are not involved in the operation of partial multiplication. These shields are simply a left-hand projection on each rod of width equal to the width of the columns which support the small figures. Single width and double width rectangular notches are cut in the projecting flaps to expose the single or double numerals as indicated by the single or double arrows respectively. Column (Rod) 1 however carries no projecting shield, but a special rod is brought into juxtaposition with the other rods at the extreme right. This rod carries a left-hand shield having eight single notches. These notches are placed so they would expose 6, 9, 2, 5, 8, 1, 4 and 7 if applied to the diagram depicted in the MONTHLY. The notches are now converted into rectangular holes by fastening a thin wire along the lip of each flap. Each wire is previously threaded through small glass beads, so that two beads are left rotatable in the single holes and three in the larger holes.

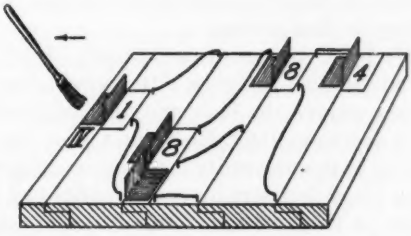
Provision is now made for shielding all the remaining unshielded small numerals. This is effected in a very simple manner. A rectangular piece of thin metal (tin is ideal) is bent about a wire slightly larger in diameter than the wires forming the left-hand boundaries of the rectangular holes. These pieces are pinched by a suitable tool around the wire to form a cylindrical hole. A cotter pin opened to 90 degrees gives the idea of the "edge-on" view of the L-shaped tin shutters. These little shutters are snapped over the supporting wires with a bead at each end of the hinge to reduce friction. An even better assembly technique would be to position them before attaching the long hinge wire. The "rods" themselves are kept in exact relationship to each other by a pair of holes part way through in the back of each which fit over studs projecting from a backing board. Now if the left-hand end of the supporting board is raised all the

rotatable shutters tilt over under the action of gravity to the L-position of stable equilibrium as viewed from bottom edge of the instrument. Every one of the small numerals is now covered.





Position 1. Rods assembled showing brush or stylus about to traverse "pin-ball" multiplier.



Position 2. Brush leaving machine having been automatically guided over zone IV by converging wires, showing product 1884. Note on tilting machine to the right all the displaced "tumbling" shutters are instantly returned to Position 1.

The lines forming the Genaille arrows are now replaced by wires bent downward at their extremities and driven into the wooden "rods." The wires are slightly above the plane of the "rods" and act as automatic guides to guide a stylus over the proper shutter as the stylus is drawn from right to left across the instrument. The shutters when pushed over by the stylus expose without am-

biguity the successive figures in the partial product and moreover the shutters operated upon fall into a position of stable equilibrium in their respective laterally inverted L-position as viewed from the bottom of the board.

The resulting partial product is immediately set up on the abacus and the Genaille board tilted to blank out all the numerals. The stylus is again drawn across the board along the row corresponding to the tens figure of the multiplier. The result is again registered on the abacus only starting this time on the tens rod. It is quite evident that the recording of successive partial products must progressively be recorded rod by rod and more and more to the left on the abacus.

OBITUARY

DAVID RAYMOND CURTISS

IN MEMORIAM

David Raymond Curtiss passed away at his home in Redlands, California on April 29 at the age of seventy-five. He had retired from his professorship of mathematics at Northwestern University in 1943 and moved to Redlands where he had spent his boyhood days. Here he built a delightful hillside home overlooking nearby orange groves and having a magnificent view of the not distant mountains. His charming flower garden attested his love of beauty and his ability to coax nature to yield her utmost for him. Unfortunately the final months of his life were marred by the serious illness of his wife who passed away with him from asphyxiation in their garage.

He was born in Derby, Connecticut, on January 12, 1878. Shortly thereafter the Curtiss family moved to California. After graduation from the Redlands high school, Raymond entered the University of California, from which he received his bachelor's degree in 1899. He spent a year teaching in a secondary school before returning to the university for a masters degree, which he received in 1901. By this time his enthusiasm for mathematics had become a controlling passion, and he went to Harvard for more advanced work. There, under the primary influence of Bocher and Osgood, he proceeded to the doctorate (1903), one of the first of a distinguished group to receive the degree under their guidance in the early years of the century. His success was rewarded by the granting of a travelling fellowship, and he went to France to study at the École Normale Supérieure in 1903–04.

After returning to this country he was instructor in mathematics at Yale for the year 1904–05. From there he went to Northwestern University where he served as head of the mathematics department for over twenty of the thirty-eight years of his active service. On leave of absence he returned to Harvard

as lecturer in 1920–21, and he taught for one summer session at California.

In 1907 he married Sigrid Eckman who died in 1941, and in 1943 he married Ruth Kneen. From the first union he has three children: John Hamilton, Margaret Eckman, and Alice Judson (now Mrs. A. W. Tucker). Raymond's scientific talents were shared by his brother, Ralph, who was a distinguished astronomer at Michigan. His son has won high recognition for his mathematical work, and his son-in-law is now chairman of the mathematics department at Princeton. A grandson is reported to have unusual mathematical ability. This is reminiscent of the famous Struve family of astronomers.

At Northwestern Curtiss was a leading member of the faculty, always highly respected by his colleagues and by the administration. As a result, he found it necessary to devote endless hours to committee work, at which he was eminently successful. Likewise the American Mathematical Society called upon him extensively for help, as member of the Council, as vice-president, as editor of the Transactions (1914–19), as editor of the Bulletin (1928–38), and as a member of many special committees. He was a charter member of the Mathematical Association of America, to which he devoted much energy as member of the finance committee and of the Board of Governors, serving as its president for two years (1935–36). For about twenty years he was on the Editorial Committee on Carus Monographs sponsored by the Association. He was a fellow of the American Association for the Advancement of Science, was vice-president (Section A) in 1921, and served on the Executive Committee for four years. He was also an editor of the Transactions of the International Mathematical Congress at Toronto (1924).

He was always interested in the teaching of mathematics, and in addition to being an exceptionally conscientious and successful teacher he wrote articles on the subject. He prepared a report for an International Commission of the Teaching of Mathematics, portions of which were published in the Bulletin of the American Mathematical Society (vol. 1911, pp. 122–137). In this MONTHLY (vol. 48, pp. 224–228) we find one of his papers, entitled *The professional interests of mathematical instructors in Junior Colleges*. His interest lay in the training of teachers, not in the methods of teaching. He always took great interest in those students who showed especial ability in mathematics. Among the many whom he encouraged to go on in advanced study, there were several whom he helped financially until they attained doctorates.

Curtiss's editorships of important mathematical publications, spanning about thirty years, clearly indicate his breadth of knowledge and his soundness of judgment. His devotion to the general interests of the mathematical world doubtless militated against his own productivity, but he did find time to write about twenty research papers and a half dozen books. His first paper, *Note on the sufficient conditions for an analytic function* (Bull. Amer. Math. Soc., vol. 8, 1902, pp. 329–331), and his second, *On the invariants of a homogeneous quadratic differential equation of the second order* (Amer. J. Math., vol. 15, 1903, pp. 355–362), were written while he was a graduate student at Harvard. His third,

Binary families in a triply connected region, with special reference to hypergeometric families (Memoirs of the American Academy of Arts and Sciences, vol. 13, 1904, pp. 2–59), was his doctoral dissertation. Somewhat closely related to these papers were three others: *Theorems converse to Riemann's on linear differential equations* (Trans. Amer. Math. Soc., vol. 7, 1906, pp. 99–106); *Sur la théorie des fonctions hypergéométriques* (Ann. École Norm. (3), vol. 22, 1906, pp. 121–143); *Relations between kindred Riemannian P and Q functions* (Bull. Amer. Math. Soc., vol. 29, 1923, pp. 154–160).

In the period from 1906 to 1911 he published four short papers in function theory: *A proof of the theorem concerning artificial singularities* (Ann. of Math. (2), vol. 7, 1906, pp. 161–164); *On certain theorems of mean value for analytic functions of a complex variable* (Ann. of Math. (2), vol. 8, 1907, pp. 118–126); *The vanishing of the Wronskian and the problem of linear dependence* (Math. Ann., vol. 65, 1908, pp. 282–298); and *Relations between the Gramian, the Wronskian, and a third determinant connected with the problem of linear dependence* (Bull. Amer. Math. Soc., vol. 17, 1911, pp. 462–467).

In 1911 he turned his attention to certain problems in the theory of equations, which resulted in a series of six papers: *An extension of Descartes' rule of signs* (Math. Ann., vol. 73, 1912, pp. 424–435); *The degree of a cartesian multiplier* (Bull. Amer. Math. Soc., vol. 20, 1913, pp. 19–26); *Extensions of Descartes' rule of signs connected with a problem suggested by Laguerre* (Trans. Amer. Math. Soc., vol. 16, 1915, pp. 350–360); *Recent extensions of Descartes' rule of signs* (Ann. of Math., vol. 19, 1918, pp. 251–278); *A mechanical analogy in the theory of equations* (Science, vol. 55, 1922, pp. 189–194); and *A note on the preceding paper* (Trans. Amer. Math. Soc., vol. 24, 1922, pp. 181–194). We do not have space to discuss any of these papers, but will remark that in Curtiss's terminology a given polynomial with real coefficients, $f(x)$, has the cartesian multiplier $f_1(x)$, a polynomial, if the Descartes rule of signs applied to the product $f(x)f_1(x)$ gives exactly the number of positive real roots of $f(x)$. He gave proofs of the existence of cartesian multipliers, and established bounds for the degree of a multiplier for a given $f(x)$; he remarked, "It is not likely that the method here given, even if improved, will have much value for numerical calculation."

In 1922, Curtiss solved a problem given in this MONTHLY by Kellogg (vol. 29, pp. 300–303) which led him to write a paper in diophantine analysis, *Classes of diophantine equations whose positive integral solutions are bounded* (Bull. Amer. Math. Soc., vol. 35, 1929, pp. 859–865).

In 1941 he published a valuable paper on *Maxima and minima of functions of two or more variables* (Mathematical Monographs, Northwestern University, vol. 1, pp. 3–43). An unpublished manuscript on *Geographical maps and their scales* was written for class-room purposes in 1943. We find that he presented a paper to the Society in 1922 entitled *On the zeros of successive polars of a binary form*, but do not find a published paper under that title.

Two non-technical papers should also be mentioned. In 1917 he was chairman of a committee which drafted a chapter on *Mathematics* for a book entitled

Science and Learning in France. And when he retired as president of the Mathematical Association he gave an instructive, and at times amusing, address on *Fashions in mathematics*, published in this MONTHLY (vol. 44, 1937, pp. 559–566).

With regard to his many papers we shall merely remark that he demonstrated a clear insight into interesting problems and presented solutions in a finished form. He did not hurry into print with embryonic ideas and subsequently add greatly to his list of titles by a succession of corrections, amendments, and extensions. He preferred to digest his subject before publication rather than to chew his cud in public.

We now turn to the books of which Curtiss was author or co-author. His first was *Analytic Functions of a Complex Variable*, which appeared in 1926 as the second of the Carus Mathematical Monographs. In this little book of 173 pages he gives a clear and concise development of the fundamental theorems of his subject, with a brief discussion of a few of its applications. In the preface he states: "Numerous citations of authorities are favorite means for an author to show his erudition—or someone else's. Perhaps this book goes to the opposite extreme and might well give a more complete statement of sources. However, the plan used in the following pages has been adopted because the author believes that the greatest service he can do for those for whom the book was intended is to induce them to supplement the reading of the present monograph by a study of accessible standard treatises on the subject."

In collaboration with a colleague, E. J. Moulton, Curtiss wrote a number of textbooks in trigonometry and analytic geometry:

Trigonometry, Plane and Spherical. 1927. 276 pp.
High School Trigonometry. 1928. 216 pp.
Analytic Geometry. 1930. 338 pp.
Exercises in Trigonometry. 1931. 223 pp.
A Brief Course in Trigonometry. 1940. 118 pp.
Essentials of Trigonometry with Applications. 1942. 174 pp.
Plane Trigonometry (for the United States Armed Forces Institute). 1943. 206 pp.
Essentials of Analytic Geometry. 1947. 259 pp.

In addition to these books of which he was co-author, Curtiss contributed a chapter on applied mathematics for a book entitled *Basic Mathematics* which appeared in 1944.

Curtiss was noted among his colleagues for his wit, often barbed, always pointed. His speech was inclined to be slow, but his mind was most active. His interests were wide and his memory superlative. Like many mathematicians, he was a lover of music. The esthetic aspects of mathematics and of the world about him appealed to him strongly; perhaps one could well say that they were dominant features of his life.

E. J. MOULTON

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MEETING OF MATHEMATICS DIVISION OF A.S.E.E.

The Mathematics Division of the American Society for Engineering Education met on June 24–26, 1953 at the University of Florida, Gainesville, Florida. Four sessions of the Division were held. The following new officers for the Mathematics Division were elected: Chairman of the Division, Dr. R. S. Burington, Chief Mathematician, Bureau of Ordnance, Navy Department, Washington, D. C.; Director of the Division, Dr. C. V. Newsom, Associate Commissioner for Higher Education, State University of New York; Member of Council of A.S.E.E., Professor H. K. Justice, University of Cincinnati.

The next meeting of the Mathematics Division of A.S.E.E. will be held June 14–18, 1954 at the University of Illinois, Urbana, Illinois. For further information write to Dr. R. S. Burington, 1834 North Hartford Street, Arlington 1, Virginia.

TRAVEL GRANTS OF THE NATIONAL SCIENCE FOUNDATION

The National Science Foundation will receive applications for a limited number of grants to assist in payment of travel expenses to the International Mathematical Congress and International Mathematical Union, Amsterdam and The Hague, Netherlands, August 30–September 9, 1954. Applications should be received by January 1, 1954. Application forms may be obtained upon request from the National Science Foundation, Washington 25, D. C.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1953 Preliminary Actuarial Examinations are as follows:

First Prize of $200

Broadwin, Emile Bernard.....Harvard University

Additional Prizes of $100

Fredkin, Donald R.....New York University
Gassner, Betty J.....New York University
Hessenthaler, Ruth.....Pembroke College
Speake, Neal M.....University of Michigan
Steiner, Lisa.....Swarthmore College
Traynor, Edwin A.....Holy Cross College
Watson, Charles B. H.....University of Toronto

Zinger, Alexis.....University of Montreal

The Society of Actuaries has authorized a similar set of nine prizes for the 1954 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. *Language Aptitude Examination.*
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. *General Mathematics Examination.*
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. *Special Mathematics Examination.*
(Finite differences, probability and statistics.)

The 1954 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 12, 1954. The closing date for applications is March 15, 1954.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS

The eighth annual Stanford University Competitive Examination in Mathematics (see this MONTHLY, vol. 53, no. 7, pp. 406–409, 1946) was held April 11, 1953 in the principal high schools of the states of California, Oregon, Washington and Arizona. This was the first year when the examination was given outside of California. Stanford put aside two scholarships of $660 each for the best papers of this competition. The following problems were proposed:

1. Bob has 10 pockets and 44 silver dollars. He wants to put his dollars into his pockets so distributed that each pocket contains a different number of dollars.
   (*a*) Can he do so?
   (*b*) Generalize the problem, considering p pockets and n dollars. The problem is the most interesting when
   $$n = \frac{(p+1)(p-2)}{2}.$$
   Why?

2. Observe that the value of
   $$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$$

is $\frac{1}{2}$, $\frac{5}{6}$, $\frac{23}{24}$ for $n=1, 2, 3$, respectively, guess the general law (by observing more values if necessary) and prove your guess.

3. Find x, y, u, and v satisfying the system of four equations

$$\begin{aligned} x + 7y + 3v + 5u &= 16 \\ 8x + 4y + 6v + 2u &= -16 \\ 2x + 6y + 4v + 8u &= 16 \\ 5x + 3y + 7v + u &= -16 \end{aligned}$$

(This may look long and boring: look for a shortcut.)

4. The four points G, H, V, and U are (in this order) the four corners of a quadrilateral. A surveyor wants to find the length $UV=x$. He knows the length $GH=l$ and measures the four angles

$$\angle GUH = \alpha, \quad \angle HUV = \beta, \quad \angle UVG = \gamma, \quad \angle GVH = \delta.$$

(*a*) Express x in terms of α, β, γ, δ, and l.
(*b*) Find some way to test the correctness of the result.
(*c*) If you had a clear plan to do (*a*) characterize it in one short sentence.

The first prize went to Mr. Peter Stauffer from the Palo Alto High School, Palo Alto, California. No paper was found worthy for the second prize. Twelve prizes of "Honorable Mention" were awarded; these persons received a book as a gift from the Department of Mathematics.

PERSONAL ITEMS

The honorary degree of Doctor of Science has been conferred upon Professor R. B. McClenon by Grinnell College; Professor McClenon has retired with the title Professor Emeritus of Mathematics.

The Thirteenth William Lowell Putnam Mathematical Competition scholarship at Harvard University has been awarded to Mr. Tai Tsun Wu of the University of Minnesota.

Alabama Polytechnic Institute announces the following: Professor W. V. Parker has been appointed Dean of the Graduate School; Professor Parker will continue to serve as the Head of the Department of Mathematics; Mr. C. W. McArthur of the University of Maryland and Dr. H. D. Sprinkle of the University of Florida have been appointed to assistant professorships.

Brown University reports the following: Professor Herbert Federer has been working under a grant from the National Science Foundation since October 1, 1952; Professor David Gale has been working under a similar grant since January 1, 1953 and Professor Bjarni Jonsson under a similar grant since February 1, 1953; Professor Gale has received a Fulbright award for research in mathematical economics to be carried on in the University of Copenhagen and the Applied Mathematics Institute of the Technical University of Denmark

during the year 1953–54.

DePaul University announces: Dr. Pasquale Porcelli of the University of Texas has been appointed to an assistant professorship; Dr. W. F. Darsow has been promoted from Lecturer to Instructor.

At Illinois Institute of Technology: Dr. R. A. Struble, previously with the Douglas Aircraft Company, Santa Monica, California, has been appointed to an assistant professorship; Dr. R. H. Oehmke, formerly a graduate student at the University of Chicago, has been appointed to an instructorship.

Michigan State College announces the following: Professor G. deB. Robinson, who is on leave of absence from the University of Toronto, has accepted a visiting professorship for the year 1953–54; Dr. J. F. Hannan of Catholic University of America has been appointed to an assistant professorship; Dr. J. W. Gaddum has resigned as mathematician with the United States Air Force to accept an instructorship at the College; Associate Professor B. M. Stewart has been promoted to a professorship; Assistant Professor L. M. Kelly has been promoted to an associate professorship; Associate Professor J. H. Bell has resigned to accept a position as supervisory mathematician with the Bendix Aviation Corporation Research Laboratories, Detroit.

Portland State Extension Center reports: Associate Professor T. S. Peterson of the University of Oregon has transferred to the Center as Associate Professor of Mathematics; Dr. W. O. Buschman has been promoted to an assistant professorship.

Princeton University announces: Professor Solomon Lefschetz, Henry Burchard Fine Professor since 1933 and Chairman of the Department of Mathematics since 1945, has retired with the title of Fine Professor Emeritus; Professor Emil Artin has been appointed Fine Professor; Associate Professor D. C. Spencer has been promoted to a professorship and Dr. J. T. Tate, Higgins Research Fellow, to an assistant professorship; Professor A. W. Tucker has been designated Chairman of the Department.

Swarthmore College announces the promotions of Dr. E. R. Mullins, Jr. and Dr. David Rosen to assistant professorships.

At Syracuse University: Assistant Professor Ruth W. Stokes has been promoted to an associate professorship; Instructor H. C. Bennett has been promoted to an assistant professorship.

The University of Texas announces that beginning September 1, 1953 there will be one mathematics department, to be known as the Department of Mathematics and Astronomy. This replaces the two former Departments of Applied Mathematics and Astronomy and of Pure Mathematics.

The University of Washington announces: Assistant Professor D. G. Chapman has been promoted to an associate professorship; Dr. M. G. Arsove has been promoted to an assistant professorship; Assistant Professor V. L. Klee of the University of Virginia has been appointed to an assistant professorship; Dr. E. A. Michael of the University of Chicago has been appointed to an assistant professorship; Dr. A. E. Livingston of the Institute for Advanced Study has

been appointed to an instructorship; Dr. R. F. Tate of the University of California has been appointed to an instructorship; Assistant Professor F. H. Brownell, Jr., has been awarded a Ford Foundation Fellowship and will spend the year 1953–54 at the Institute for Advanced Study.

Assistant Professor A. G. Anderson of Oberlin College has been appointed to an assistant professorship at Duquesne University.

Mr. R. E. Barlow, formerly a student at Knox College, has been appointed to an assistant instructorship at the University of Oregon.

Dr. J. D. Baum, previously a student at Yale University, has been appointed to an instructorship at Oberlin College.

Assistant Professor H. M. Beatty of Ohio State University has retired.

Professor E. T. Bell of California Institute of Technology has retired with the title of Professor Emeritus.

Dr. Eleazer Bromberg of the Office of Naval Research, Washington, D. C., has been appointed Assistant Chief of the Computing Center, Institute of Mathematical Sciences, New York University.

Dr. R. S. Burington, chief mathematician of the Bureau of Ordnance, Navy Department, and head of the Evaluation and Analysis Group of the Bureau of Ordnance, has been named Special Assistant to the Director of Research and Development, Bureau of Ordnance, Navy Department, Washington, D. C.

Mr. R. G. Buschman of McNeese State College has accepted an appointment as a fellow at the University of Colorado.

Mr. S. C. Cobb, formerly at Phillips Academy, Andover, Massachusetts, has accepted a position at St. John's School, Houston, Texas.

Associate Professor Rufus Crane of Ohio Wesleyan University has retired.

Mr. T. H. Dewey, previously a research assistant at Los Alamos Scientific Laboratory, New Mexico, has a position as a mathematician at the Radiation Laboratory of the University of California.

Mr. R. C. DiPrima, who has been a part-time instructor at Carnegie Institute of Technology, has received an appointment as a research associate at Massachusetts Institute of Technology.

Mr. M. E. Drummond, formerly a research assistant at the University of Oklahoma, has accepted a position as a mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Mr. Anderson Duggar, Jr., previously a student at the University of Detroit, is now President of the Equitable Engineering Company, Detroit, Michigan.

Mr. G. W. Fairchild, who has been associated with Northrop Aircraft Incorporated, Hawthorne, California, has a position as a research analyst with Bendix Computer Division, Los Angeles, California.

Dr. J. M. G. Fell has been appointed to an instructorship at California Institute of Technology.

Dr. A. M. Feyerherm of Iowa State College has been appointed to an assistant professorship at Kansas State College.

Assistant Professor N. J. Fine of the University of Pennsylvania has been

promoted to an associate professorship; he will spend the academic year 1953–54 at the Institute for Advanced Study on a National Science Foundation Postdoctoral Fellowship.

Dr. C. D. Firestone has accepted a position as a mathematician at the Applied Physics Laboratory, Johns Hopkins University.

Dr. H. D. Friedman, formerly a graduate student at Pennsylvania State College is employed by the General Electric Company, Syracuse, New York.

Mr. E. P. Graney of the Willow Run Research Center, University of Michigan, has been promoted to the position of Research Associate.

Dr. E. J. Gumbel of Columbia University has been promoted to an adjunct professorship; during the summer term he served as Visiting Professor at the Free University of Berlin.

Professor Georgia M. Haswell of Kansas Wesleyan University has been appointed Dean of Women at Pfeiffer Junior College, Misenheimer, North Carolina.

Mr. R. E. Horton, who has been in the United States Air Force, has returned to his position as an instructor at Los Angeles City College.

Associate Professor H. T. Karnes of Louisiana State University has been promoted to a professorship.

Instructor M. S. Klamkin of Brooklyn Polytechnic Institute has been promoted to an assistant professorship.

Miss Rachael La Roe, head of the Department of Mathematics and Physics at Grand Canyon College, has been appointed Associate Professor of Mathematics and Physics at Mary Hardin-Baylor College.

Mrs. Helen H. Meek, formerly at the David Taylor Model Basin, Washington, D. C., has a position as a mathematician at the Institute for Numerical Analysis, National Bureau of Standards.

Mr. V. A. Miculka of Frank Phillips College, Borger, Texas, has accepted a graduate assistantship at the University of Oklahoma.

Assistant Professor G. W. Morgan, Graduate Division of Applied Mathematics, Brown University, has been promoted to an associate professorship.

Mr. H. W. Morrow, Jr. of the University of Florida is engaged as a mechanical engineer with the Texas Automatic Sprinkler Company, Houston, Texas.

Assistant Professor D. J. Myatt of Antioch College has accepted a position with the Atlantic Research Corporation, Alexandria, Virginia.

Dr. W. R. Orton of Oberlin College has been appointed to an assistant professorship at the University of Arkansas.

Assistant Professor F. D. Parker of St. Lawrence University has been appointed to an associate professorship at Clarkson College of Technology.

Assistant Professor W. D. Peeples, Jr., of Howard College has been promoted to an associate professorship.

Associate Professor P. I. Peters of Union College has accepted a position as an actuarial assistant with the Commonwealth Life Insurance Company, Louisville, Kentucky.

Dr. Harry Polachek of the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland, has accepted a position as Head of the Applied Mathematics Laboratory, David Taylor Model Basin, Washington, D. C.

Dr. Irene Price, previously at Holloman Air Force Base, New Mexico, has a position as a mathematician at White Sands Proving Ground, Las Cruces, New Mexico.

Dr. L. M. Rauch, who has been at the Research Center, University of Michigan, is engaged now as a research mathematician with Consolidated Aircraft Company, Fort Worth, Texas.

Dr. L. J. Reed, who has recently retired from his position as Vice-President of the Johns Hopkins University and Hospital, has been appointed President of the University.

Professor B. L. Schwartz of Duquesne University has a position as Principal Electrical Engineer at Battelle Memorial Institute, Columbus, Ohio,

Dr. Raymond Sedney, formerly with Douglas Aircraft Company, Santa Monica, California, has accepted a position at the Exterior Ballistics Laboratory, Aberdeen Proving Ground, Maryland.

Professor L. L. Smail of Lehigh University has retired.

Lt. Malcolm Smith has been assigned to the Wright Air Development Center as a mathematician.

Assistant Professor J. S. Stubbe of Clark University has been promoted to an associate professorship.

Assistant Professor W. J. Thron of Washington University has been promoted to an associate professorship.

Dr. A. J. Tingley of the University of Nebraska has been appointed to an assistant professorship at Dalhousie University, Halifax, Nova Scotia, Canada.

Mr. H. G. Tucker, formerly of Rutgers University, has a position as a graduate research statistician at the University of California.

Dr. L. E. Ward, previously a graduate fellow at Tulane University, has been appointed to an instructorship at the University of Nevada.

Dr. F. C. Warner of the University of Buffalo has been appointed to an assistant professorship at St. Lawrence University.

Mr. J. A. Weiss, who has been a research fellow at Ohio State University, has accepted a position with the Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. Hazel S. Wilson has been appointed to an associate professorship at Jacksonville State Teachers College, Alabama.

Assistant Professor C. N. Yancey of Francis T. Nicholls Junior College has a position as a computer with the Shell Oil Company, Arcadia, Louisiana.

Professor W. F. C. Arndt of the University College of the Orange Free State, Union of South Africa, died on June 17, 1951.

Professor A. D. Michal of California Institute of Technology died on June 14, 1953; he had been a member of the Association for thirty-three years.

Miss Edna M. Norskog of Illinois State Normal University died on May 14, 1953.

Reverend D. A. Steele, professor of mathematics at Fordham University, died on June 18, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 78 persons have been elected to membership by the Board of Governors on applications duly certified.

N. E. ALBRECHT, M.A. (Minnesota) Asst. Professor, Hamline University.

R. F. ANASTASIO, B.S. (C.C.N.Y.) Aeronautical Research Scientist, Ames Aeronautical Laboratory, Moffett Field, Calif.

MRS. RUTH E. BERARD, B.S. (Medical Evangelists) Medical Technologist, St. Catherine's Hospital, East Chicago, Ind.

P. M. BERRY, Student, University of Oklahoma.

DAVID BLACKWELL, Ph.D. (Illinois) Professor, Howard University.

D. H. BLANKSTEEN, B.S. (C.C.N.Y.) Office Manager, E. & W. Blanksteen, New York City.

R. O. BLUMMER, JR., M.S. (N. Texas S. C.) Asso. Mathematician, Vitro Corporation of America, Silver Spring, Md.

ELEAZER BROMBERG, Ph.D. (N.Y.U.) Asst. Chief, Computing Center, Institute of Mathematical Sciences, New York University.

MORTON BROWN, Student, University of Wisconsin.

EUGENE BUTKOV, Student, University of British Columbia.

M. L. CLINNICK, Student, University of California.

R. F. DENNEMEYER, M.A. (U.C.L.A.) 715C West Queen Street, Inglewood, Calif.

ROLAND DEWIT, Room 914, Central Y.M.C.A., Dayton 2, Ohio.

V. J. DIGRICOLI, B.S. (Manhattan) Mathematician, International Business Machines Corporation, New York City.

CHESTER DOMORACKI, B.S. (Alliance) 25 Temple Street, Haverhill, Mass.

T. H. ELFERT, B.S. (St. Norbert) 317 14th Avenue, Green Bay, Wis.

M. P. EMERSON, Ph.D. (Illinois) Asst. Professor, Harpur College.

R. J. ENGERT, M.S. (Illinois) Grad. Student, University of Illinois.

P. C. FIFE, B.A. (California) Grad. Student, University of California.

EDWARD FLANDERS, B.A. (Oklahoma A. & M.) 1502 West Admiral Road, Stillwater, Okla.

F. A. FLORIO, B.A. (La Salle) Physicist, Frankford Arsenal, Philadelphia, Pa.

ALLEN FOX, Student, Brooklyn College.

RICHARD GABEL, M.A. (Columbia) Grad. Student, American University.

I. B. GOLDBERG, Student, New York University.

M. J. GOLDSTEIN, Student, Memphis State College.

A. J. GUNTHER, M.A. (Columbia) Teacher, Avon Old Farms, Avon, Conn.

W. J. HALLMARK, M.A. (Texas) Professor, San Antonio College.

W. J. HARDELL, M.S. (Michigan S. C.) Grad. Assistant, Michigan State College.

A. R. HARVEY, Ph.D. (Harvard) Assoc. Professor, San Diego State College.

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SEVERIN LOW, Student, McGill University.

SAUL MANDEL, Student, University of Oklahoma.

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ESSOR MASO, B.A. (U.C.L.A.) Supervisor Scientific Computer, Hughes Aircraft Company, Culver City, Calif.

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C. M. SOMMERFIELD, Student, Brooklyn College.

JEAN STREMP, B.A. (Seton Hill) Student, Seton Hill College.

J. P. THOMSEN, Student, American University.

HAROLD WEITZNER, Student, University of California.

N. A. WHEELER, Student, Reed College.

L. G. WOODBY, Ph.D. (Michigan) Asso. Professor, State Teachers College, Mankato, Minn.

THE MARCH MEETING OF THE METROPOLITAN NEW YORK SECTION

The twelfth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Teachers College, Columbia University, New York City, on March 28, 1953. Professor W. H. Fagerstrom, Collegiate Vice-Chairman, presided at the morning session, and Professor L. F. Ollmann, Chairman of the Section, presided at the afternoon session.

One hundred thirty-two persons attended the meeting, including the following eighty-four members of the Association:

R. G. Archibald, Winifred A. Asprey, L. F. Babcock, C. Y. Bartholomew, J. S. Bergen, Samuel Borofsky, C. B. Boyer, A. D. Bradley, A. B. Brown, Jewell H. Bushey, A. J. Carlan, Audrey M. Carlan, P. L. Chessin, Charles Clos, P. J. Cocuzza, H. R. Cooley, T. F. Cope, W. H. H. Cowles, Mary P. Dolciani, Jesse Douglas, Brother E. I. Duggan, J. N. Eastham, Samuel Eilenberg, W. H. Fagerstrom, H. F. Fehr, William Forman, R. M. Foster, E. T. Frankel, Leona Freeman, Brother Bernard Gerald, B. P. Gill, Bernard Greenspan, G. C. Helme, L. J. Herbach, Arthur Herskowitz, E. Marie Hove, T. R. Humphreys, O. J. Karst, E. R. Kiely, H. S. Kieval, A. E. Kinney, M. S. Klamkin, Morris Kline, Charles Koren, David Kotler, H. C. Kranzer, R. A. C. Lane, C. H. Lehmann, M. E. Levenson, D. M. MacEwen, J. R. Macy, J. H. Manheim, May H. Maria, K. A. McGown, Brother J. G. McKenna, F. H. Miller, Jack Minker, Morris Morduchow, Eugene Odin, L. F. Ollmann, C. F. Pinzka, J. J. Quinn, J. K. Reckzeh, M. R. Reeks, Selby Robinson, I. H. Rose, N. J. Rose, M. F. Rosskopf, H. D. Ruderman, J. P. Russell, D. A. Russo, John Salerno, Charles Salkind, Aaron Shapiro, E. I. Shapiro, James Singer, Sister M. Anita, E. R. Stabler, R. L. Swain, N. Y. Tang, H. E. Wahlert, Etta A. Waite, M. E. White, R. C. Yates.

The following officers were elected for the year 1953–54: Chairman, Professor W. H. Fagerstrom, City College of the City of New York; Collegiate Vice-Chairman, Professor H. F. Fehr, Teachers College, Columbia University; High School Vice-Chairman, Mr. Charles Salkind, Samuel Tilden High School, Brooklyn; Secretary, Professor E. Marie Hove, Hofstra College; Treasurer, Mr. Aaron Shapiro, Midwood High School, Brooklyn.

At the business meeting, the following report on the activities of the Committee on Contests and Awards was given by its chairman, Professor W. H. Fagerstrom:

The Committee on Contests and Awards of the Metropolitan New York Section reported that 339 schools had registered for the fourth annual contest and that these schools had requested 9,672 copies of the contest booklets. In addition to these schools, there are three units operating independently using the Metropolitan New York Section's questions. These units are centered at 1) The University of Oregon, 2) The University of British Columbia, and 3) The Board of Education of Buffalo, N. Y. These three units have requested a total of 2,300 booklets. The 339 schools listed above are from 29 states and 2 Canadian provinces. The contest will be held on May 14, 1953.

Dr. Paul Bulger, Assistant Provost of Columbia University, welcomed the people at the meeting, and then the following papers were presented:

1. *Mathematics and the liberal arts student*, by Professor Morris Kline, New York University.

The practice in American colleges and universities of offering college algebra and trigo-

nometry to students who do not intend to use mathematics in later life was severely criticized. These courses have no cultural content and consist of dry, unmotivated, and purely technical procedures which students learn to repeat in parrot-like fashion. Instead of such courses, the speaker described and recommended a course which presents basic mathematical concepts in the cultural context in which these concepts arose and which includes a presentation of the significance of these concepts for modern civilization and culture. Such a course is being tried at the Washington Square College of New York University.

2. *Theory and methods of electronic digital computation*, by Mr. J. H. White, Jr., International Business Machines, New York City, introduced by the Secretary.

This paper states what one can expect of computing machines and particularly why they are important for scientific and engineering problems. Each of IBM's three types of electronic calculators is described in terms of the five elements of logical design: input, output, storage, arithmetic unit, and control unit. They are used in industry, government, and education. Twenty-six colleges and universities now have IBM computing installations. Some of the types of mathematical and physical problems readily solved by these machines are mentioned with emphasis on a problem of aircraft propeller design involving solution of two simultaneous fourth-order differential equations reduced to a set of forty simultaneous linear equations in forty unknowns.

3. *Trends in mathematics: the elementary school*, by Dr. Laura K. Eads, Board of Education of the City of New York, introduced by the Secretary.

The elementary school mathematics program in New York City, designated as *Developmental Mathematics*, is moving toward helping children to think mathematically: to discover mathematical relationships, to derive their own generalizations, to develop independence in problem-solving through estimation and computation without the use of paper and pencil. This program emphasizes the development of mathematical concepts, facts, and processes through levels of learning, engaging in experience, using representative materials, thinking through mathematical relationships, written computation. The mathematics is learned sequentially from the first grade. Understanding of the mathematical meaning or structure of numbers and processes, and of measurement is developed before skill in computation is emphasized.

4. *Curriculum trends in high school mathematics*, by Mr. Max Peters, Long Island City High School, New York City, introduced by the Secretary.

The most significant factor affecting the mathematics curriculum in the high schools is the fundamental change in student composition. There is grave danger that the traditional academic mathematics courses will be seriously weakened in the attempt to reach the mediocre and poor student. The development of second track courses can help to check this tendency. The two major movements in the academic mathematics high school curriculum are greater emphasis upon meaning and more systematic attempts at integration. Complicated manipulation in algebra has been de-emphasized in favor of rationalization of operations. The postulational structure of plane geometry has been stressed. Units of coordinate geometry have been included in plane geometry courses and progress has been made in developing a fused course in the eleventh year combining intermediate algebra and trigonometry.

5. *Topology: Its relation to other branches of mathematics*, by Professor Samuel Eilenberg, Columbia University.

H. S. KIEVAL, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-sixth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Colorado, Boulder, Colorado, on April 17 and 18, 1953. Professor B. W. Jones, Chairman of the Section, presided at all the sessions.

Of the eighty-five persons who registered, the following fifty-five were members of the Association:

C. F. Barr, D. L. Barrick, B. C. Bellamy, W. E. Briggs, J. R. Britton, R. K. Butz, F. M Carpenter, Sarvadaman Chowla, G. S. Cook, F. W. Donaldson, W. E. Dorgan, F. N. Fisch, C. A. Grimm, Arnold Grudin, H. T. Guard, R. R. Gutzman, Marian S. Gysland, C. L. Harbison, Leota C. Hayward, I. L. Hebel, Ruth I. Hoffman, LeRoy Holubar, P. F. Hultquist, J. A. Hurry, C. A. Hutchinson, B. W. Jones, A. J. Kemper, Claribel Kendall, J. S. Leech, D. C. B. Marsh, Jr., Garner McCrossen, H. C. McKenzie, E. B. McLeod, Jr., W. E. Mientka, W. K. Nelson, Greta Neubauer, D. K. Parks, Lily B. Powell, O. M. Rasmussen, O. H. Rechard, A. W. Recht, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, K. H. Stahl, P. O. Steen, E. L. Swanson, C. W. Thomson, E. P. Tovani, E. L. Vanderburgh, V. J. Varineau, W. W. Varner, J. F. Wagner.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor M. L. Madison, Colorado Agricultural and Mechanical College; Vice-Chairman, Professor Nathan Schwid, University of Wyoming; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The program of papers for the meetings was as follows:

1. *Some results in number theory using a partial summation method*, by Mr. W. E. Briggs, University of Colorado.

In the classical proofs of theorems concerning the representation of primes by binary quadratic forms, it is necessary to use facts about the continuity, differentiability, and behavior as $s \to 1^+$ of the series $\sum (ax^2+2bxy+cy^2)^{-s}$ and other series similar to it. The summation is extended over all x, y which make the form prime to $2D$, where $D=b^2-ac$, and which satisfy certain other conditions if $D>0$. These facts can all be proved simply by estimating the sum as $\sum_{n=1}^{\infty} n^{-s}[T(n) - T(n-1)]$, where $T(n)$ is the number of lattice points within $ax^2+2bxy+cy^2=n$ which make the form prime to $2D$ and satisfy the other conditions if $D>0$.

2. *Polynomials associated with matrices*, by Professor R. K. Butz, Colorado Agricultural and Mechanical College.

Notation was developed to handle the matric equation $AX=XB$, where A and B are specified matrices of order n and m, respectively, defined over an arbitrary field F, and X is to be determined in terms of parameters using only those operations with respect to which F is closed. The approach to the problem was that given by W. V. Parker (*The matrix equation* $AX=XB$, Duke Mathematical Journal, vol. 17, no. 1, 1950, p. 43).

3. *Some topics in the theory of numbers*, Professor Sarvadaman Chowla, University of Colorado.

4. *An approximation method in certain nonlinear boundary value problems*, by Professor Nathan Schwid, University of Wyoming.

In the differential equation of heat conduction,

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right),$$

the physical quantities c, the specific heat, and K, the thermal conductivity, are usually considered constant. When these quantities are more realistically regarded as linear functions of the temperature u, the equation is nonlinear. A method of approximation to the solution of the equation, subject to suitable boundary conditions, is here discussed. The method is applicable to situations where the solution for constants c and K is in the form of an infinite series of orthogonal functions each term of which has an exponential factor with negative exponent.

5. *Linear diophantine equations and additive number theory*, by Professor Emeritus A. J. Kempner, University of Colorado. (By invitation.)

We know little about solutions of linear diophantine equations with prescribed restrictions, except for such cases as all solutions positive, or all solutions bounded, *etc.* It is interesting that large groups of problems in the additive number theory can be paraphrased into problems in linear diophantine equations with a certain type of restriction on the solutions. Thus,

$$1x_0 + 3x_1 + 5x_2 + \cdots + (2m+1)x_m + \cdots = n$$

has a solution $4 \geqq x_0 \geqq x_1 \geqq \cdots \geqq x_m > 0$ for all positive integral n; $28 = 4 \cdot 1 + 4 \cdot 3 + 1 \cdot 5 + 1 \cdot 7$; but has a solution $3 \geqq x_0 \geqq x_1 \geqq \cdots \geqq x_m > 0$ when and only when $n \neq 4^s(8t+7)$. The paraphrase is based on the simple fact: Given a set, finite or infinite, of positive integers $a_0, a_1, \cdots, a_m, \cdots$, with $a_{m+1} > a_m$ (for convenience), and the set $d_0 = a_0$, $d_1 = a_1 - a_0$, $d_2 = a_2 - a_1, \cdots$, then the two statements are equivalent: (a) a given positive integer n is the sum of at most k elements a_m (repetition allowed), and (b) the diophantine equation $d_0x_0 + d_1x_1 + \cdots + d_ma_m + \cdots = n$ has a solution $k \geqq x_0 \geqq x_1 \geqq \cdots \geqq x_m > 0$. Application is made to Pythagorean numbers ($a^2 + b^2 = c^2$), the formulation of Fermat's theorem ($a^n + b^n = c^n$), to polygonal and pyramidal numbers, to such equations as $3^x = 2^y + 1$, or $3^x = y^3 + z^3$, *etc.* Emphasis is placed on the Waring-Hilbert theorem on powers, and the Waring-Kamke theorem on polynomials with rational coefficients and integral function values for integral argument values. The Waring-Kamke theorem contains the Waring-Hilbert theorem as a very special case. The paraphrase of the Waring-Kamke theorem may be stated as follows: Let $a_0 = 1, a_1, \cdots, a_m, \cdots$ (positive integers, increasing) be the elements of an arithmetic progression of order k, and let the (positive) first differences $d_0 = a_0$, $d_1 = a_1 - a_0$, $d_2 = a_2 - a_0, \cdots$ also be increasing (for convenience). Then there exists a positive integer $N = N(k; a_0, \cdots, a_k)$, independent of n and of $a_{k+1}, a_{k+2}, \cdots$, such that $d_0x_0 + d_1x_1 + \cdots + d_mx_m + \cdots = n$ has a solution $N \geqq x_0 \geqq x_1 \geqq \cdots \geqq x_m > 0$ (n any positive integer). If $a_0 > 1$, there exists an N as above, and a positive integer $L = L(k; a_0, \cdots, a_k)$ such that in every interval of length L there is at least one positive integer n for which the equation has a solution $N \geqq x_0 \geqq \cdots \geqq x_m > 0$.

The preceding paper was the invited address for the evening session following the customary banquet.

6. *On sets of quasi-conjugate matrices*, by Professor V. J. Varineau, University of Wyoming.

A set of quasi-conjugate matrices is defined by removing the commutativity restriction from the usual definition of conjugate matrices. Thus, a set, $A_1, A_2, \cdots, A_n$, of $n \times n$ matrices over a field $\mathfrak{F}$ is quasi-conjugate if the matrices A_i are similar and if $I|xI - A_1| = (xI - A_1)(xI - A_2) \cdots (xI - A_n)$. Elementary properties of such sets are presented and conjectures about general existence theorems are made.

7. *A problem from the* MONTHLY: *Number 4479*, by Professor Emeritus A. J. Kempner, University of Colorado.

Making use of elementary properties of the roots of unity, it is shown that $a_1, a_2, \cdots, a_j, \cdots$ (all $\neq 0$) can be determined so that each $\sum_{j=1}^{\infty} a_j^k$, $k = 1, 2, 3, \cdots$, converges (conditionally) to zero. Each a_j is of the form $\gamma_j \cdot \epsilon_j$, γ_j real, ϵ_j some root of unity.

8. *The content and method for a general mathematics course for adults*, by Miss Ruth I. Hoffman, Byers Junior High School, Denver, Colorado.

9. *The rapid growth of numerical analysis since 1943 and the challenge it offers to the university teacher*, by Mr. W. W. Varner, University of Colorado.

The mushrooming of numerical analysis and its inherent problem of error consideration has greatly increased the need of all teachers of mathematics, engineering, and the sciences to be meticulous in demanding that problem solutions be written in such a manner that the error or uncertainty in every result be clearly and unmistakably given. Certain aspects of this problem were discussed.

10. *On the improvement of service courses in freshman mathematics*, by Professor I. L. Hebel, Colorado School of Mines.

An outline is presented of a revised approach to the teaching of freshman mathematics adopted at Colorado School of Mines. The traditional sequence of topics is replaced by a unification that stresses the analytic geometry viewpoint throughout and that maintains the necessary rigor of a pre-engineering mathematics course. Principal results of the initial trial of the plan include increased student interest, improved faculty instruction, and a better prepared student for sophomore courses. The author and his staff feel that such a curriculum is a forward step in the solution of the perplexing freshman teaching problem.

11. *Mathematics used in university departments other than mathematics or engineering*, by Professor O. M. Rasmussen, University of Denver.

A report was presented on a part of a study using the methods of textbook analysis, interviews, and questionnaires to determine those mathematical skills and concepts that are desirable as preparation for non-mathematics course work for university students not majoring in mathematics or the physical sciences. The mathematics needed for a vast majority of these students is quite elementary and many of the students do not possess sufficient arithmetical maturity to enable them to gain maximum benefit from a large number of university courses. An understanding of elementary statistics is needed in many courses throughout the university.

J. R. BRITTON, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Texas Christian University, Fort Worth, Texas, on April 24–25, 1953. Professor C. B. Wright, Chairman of the Section, presided at the sessions. Professor L. R. Ford, who was an invited guest, contributed much to the success of the meeting.

There were one hundred eight persons in attendance, including the following sixty-two members of the Association:

T. A. Abouhalkah, R. C. Ailara, A. W. Ashburn, A. V. Banes, Ina M. Bramblett, H. E. Bray, Myrtle C. Brown, M. L. Coffman, L. A. Colquitt, J. V. Cooke, Don Cude, F. W. Donaldson, G. H. Dubay, L. K. Durst, Terrell Ellis, L. R. Ford, Gordon Fuller, R. L. Glass, Blanche B. Grover, W. T. Guy, Jr., E. H. Hanson, E. A. Hazelwood, E. R. Heineman, Fay H. Johnson, Ruth Kissel, E. C. Klipple, H. A. Luther, Hazel L. Mason, Lida B. May, Dorothy McCoy, W. K. McNabb, V. A. Miculka, Harlan C. Miller, B. C. Moore, E. D. Mouzon, Jr., C. A. Murray, Albert Newhouse, Bob Parker, H. C. Parrish, C. J. Pipes, C. B. Rader, Sr., L. W. Ramsey, Dorothy L. Rees, C. L. Riggs, Virginia E. Roberts, C. A. Rogers, R. Q. Seale, C. R. Sherer, D. P. Shore, Sister Mary of Perpetual Help, D. W. Starr, W. G. Stokes, W. W. Taylor, Earl Thomas, F. E. Ulrich, R. S.

Underwood, Margaret M. Welch, Mabel Williams, H. A. Wood, A. W. Wortham, C. B. Wright, Martin Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Professor L. A. Colquitt, Texas Christian University; Vice-Chairman, Professor E. A. Hazelwood, Texas Technological College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The program consisted of the following papers:

1. *The summation of certain trigonometrical series by undergraduate methods*, by Professor E. C. Klipple, Agricultural and Mechanical College of Texas.

Certain trigonometric series can be summed without the use of complex numbers or general convergence theorems. In particular, the series $\cos x+\frac{1}{2}\cos 2x+\frac{1}{3}\cos 3x+\cdots$ can be summed by elementary methods.

2. *Partial difference equations of elliptic type*, by Mr. Horace Flatt, Rice Institute, introduced by the Secretary.

Several properties of partial difference equations of elliptic type are discussed in this paper and in particular analogies between the theory of differential and difference equations are pointed out. The existence and uniqueness of a solution for the first boundary value problem for the general elliptic equation of second order is shown.

3. *The first two weeks of calculus*, by Professor L. R. Ford, University of Mississippi.

The speaker proposed the early introduction of the notions of upper and lower bounds. These lead naturally to the concepts of area, length of a curve, and the like. Further, they provide a reasonably simple approach to the ideas of continuity and of limit.

4. *A technique for simultaneous quadratics*, by Professor R. S. Underwood, Texas Technological College.

The usual elimination methods of algebra are almost hopelessly inadequate to find common real solutions of simultaneous quadratic equations when the number of unknowns exceeds two or three. However, many problems of this nature are solved easily by algebraic methods which are guided by the graphs of extended analytic geometry. For simplicity of presentation the examples chosen to illustrate the method involve only four or five unknowns, but this restriction is not necessary.

5. *The analytic and geometric definitions of the trigonometric functions*, by Professor E. R. Keown, Agricultural and Mechanical College of Texas, introduced by the Secretary.

The purpose of this paper is to derive the Euler formula $e^{ix}=\cos x+i\sin x$ and to show explicitly the connections between the analytic definition of the trigonometric functions as power series and their geometric definitions as ratios of line segments. The argument consist in showing that e^x as defined by the series is really a power and that e^{ix} is a complex number of modulus one for every real x whose real and imaginary parts are the $\cos x$ and the $\sin x$ when a proper choice of angular measure is made.

6. *Certain polynomial approximations*, by Mr. R. M. McLeod, Rice Institute, introduced by the Secretary.

Two theorems dealing with approximation of functions holomorphic and never zero in a

simply-connected region D and on its analytic boundary C are considered. The first gives a sequence of polynomials with all zeros on C which approximates the function in D so that the convergence is geometric in compact subsets of D. The second shows that in general the approximation can not be much improved since the only functions which can be approximated by polynomials with all zeros in a half-plane are certain entire functions.

7. *A solution of the differential equation for a stationary mass of gas*, by Professor J. T. Hurt, Agricultural and Mechanical College of Texas, introduced by the Secretary.

Previously no general solution for the problem of a gas at constant temperature held together by gravitation has been found. It is shown that the introduction of a temperature variation permits a general solution to be found.

8. *Entire functions associated with a class of Riemann surfaces*, by Mr. H. C. Filgo, Jr., Rice Institute, introduced by the Secretary.

The class W_k of Riemann surfaces consists of those surfaces obtainable by taking the surface of $z=\arcsin w$, with its branch points displaced along the real axis, and repeatedly attaching logarithmic ends along cuts placed symmetrically with respect to the real axis until there are k such extensions. The surfaces in W_k are parabolic, and correspond one-one with functions $f(z)$ satisfying $f'(z)=e^{p(z)}\prod_{n=-\infty}^{\infty} E(z/b_n, q)$ where q depends on the monotone increasing sequence of real numbers b_n, and $p(z)$ is a polynomial with real coefficients. The number k is specified by q and the degree of $p(z)$.

9. *Definitions and the questioning mind*, by Professor W. T. Guy, Jr., University of Texas.

A plea was entered for more careful definitions of mathematical terms used in the classroom. Poor definitions which give rise to "sloppy thinking" were contrasted with better definitions. It was suggested that definitions poorly phrased and improperly introduced fail to give an atmosphere conducive to the development of a questioning mind.

10. *Converses of theorems*, by Dr. L. K. Durst, Rice Institute.

Several meanings of the word "converse" were discussed and examples given to illustrate them. Particular attention was given to theorems of the form A and B imply C and 31 variants of such theorems were constructed, each of which may claim the title "converse" in one way or another. The structure of this class of variants was described and the relationship of that structure to the logical nature of the original theorem was indicated.

11. *A digression on "The Things I Should Have Done, I Did Not Do,"* by Miss Hazel L. Mason, Junior High School, Grand Prairie.

Digressing on a statement by Professor R. E. Langer, "It (teaching) calls at each level for especial qualifications" (this MONTHLY, vol. 59, 1952, page 446), this paper dealt with the special needs of public school teachers in addition to adequate training in mathematics and the usual professional courses. That practical means exist for meeting these needs were pointed out. The purposes for meeting such needs were discussed: to engender better relations with public schools and to assist in solving the problem of maintaining adequate and competent mathematics teaching personnel in public schools.

12. *The importance of mathematics to the petroleum engineers*, by Mr. T. A. Abouhalkah, University of Texas.

Being a motivating element of any field of engineering, mathematics is an important require-

ment which must be met to enable a professional engineer to digest in a sufficient manner the statements and solutions of many problems of new books and articles written concerning the physical and chemical properties of oil and gas. Many mathematical formulae and equations are shown and solved, all of which are pertaining to petroleum engineering problems. The author has in mind an effective course of mathematics combining the working essence of formulae and their manipulation compiled, digested, and taught for the particular need of engineers. A lack of mathematics is the fault of the school, not the students. Engineering has become too exact to depend upon sketchy memory and doubt and manuals solicited from large manufacturers.

13. *The content of business mathematics*, by Professor Bob Parker, Texas Technological College.

There are three courses that ordinarily come under the heading of business mathematics: a first semester course, mathematics of finance, and statistics. The first course should be college algebra and not a version of business arithmetic since the student will need algebra in his later courses. In the mathematics of finance the line diagram and equation of value should be stressed. Statistics, other than an introductory course, should not be a business subject because the students lack the mathematical background for understanding the subject.

14. *Recent developments in the certification of teachers in Texas*, by Professor E. A. Hazlewood, Texas Technological College.

This paper deals with general developments in the teacher training field in Texas subsequent to the enactment of the Gilmer-Aikin Bills in 1949. Results of the statewide study of certification sponsored by the Texas Education Agency and other interested groups are reviewed. The provisions of the Certification Bill (H. B. 496) now before the Texas Legislature are discussed.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twenty-seventh meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Carnegie Institute of Technology, Pittsburgh, Pennsylvania, on May 2, 1953. Dr. Morris Ostrofsky, Chairman of the Section, presided at the morning session, while Professor David Moskovitz of the host institution presided in the afternoon.

There were seventy-five persons present including the following forty-five members of the Association:

Thomas Bauserman, Helen Calkins, J. G. Christiano, A. B. Cunningham, H. A. Davis, R. C. DiPrima, W. A. Dolid, Esther S. Dunkelberger, R. D. Edwards, Mary A. Goins, Hunter Hardman, Evan Johnson, Jr., R. P. Johnson, F. E. Justis, George Laush, C. E. Lemke, R. G. McDermot, David Moskovitz, L. T. Moston, B. H. Mount, Jr., Pauline E. Mount, J. H. Neelley, Ruth E. O'Donnell, E. G. Olds, Morris Ostrofsky, W. J. Pervin, I. D. Peters, G. P. Rheubinall, Louis Sacks, E. A. Saibel, B. L. Schwartz, D. H. Shaffer, I. M. Sheffer, Sister M. Deborah, F. H. Steen, J. K. Stewart, E. A. Sturley, T. T. Tanimoto, F. H. Taylor, J. S. Taylor, Margaret O. Taylor, Jean E. Teats, M. L. Vest, E. B. Weinberger, E. A. Whitman.

It was voted to support an autumn meeting designed to interest high school teachers particularly.

Officers elected to serve from June 1953 to June 1955 were: Chairman, Professor F. H. Steen, Allegheny College; Secretary-Treasurer, Dean L. T. Moston, Waynesburg College; Executive Council, President B. C. Patterson,

Washington and Jefferson College, and Professor B. P. Hoover, Carnegie Institute of Technology.

The following resolution was adopted:

Whereas the Allegheny Mountain Section of the Mathematical Association of America is deeply concerned regarding the effects on the morale of scientists in government caused by the recent dismissal of Dr. Alfred V. Astin from his post as Director of the National Bureau of Standards; and whereas no good reason for this action has been given publicity; be it resolved that the Secretary of this Section address a letter to the Secretary of Commerce enclosing a copy of this Resolution and requesting him to appoint a committee of qualified scientists to review the work of the National Bureau of Standards with regard to the battery additive AD-X2, and to authorize prompt publication of the findings of this committee. Further, be it resolved, that the Secretary of this Section address a letter to the Secretary of the Mathematical Association of America enclosing a copy of this resolution and of the above-mentioned letter, and urging prompt official action by the Mathematical Association of America with regard to the Astin issue.

Papers presented were:

1. *The dual method of solving the linear programming problem*, by Mr. C. E. Lemke, Carnegie Institute of Technology.

A new method of solving the linear programming problem, *i.e.* maximize $\sum_{i=1}^{m} b_i x_i$, where the x_i's are subject to $\sum_{i=1}^{m} a_{ij} x_i$, $j = 1, \cdots, n$, was discussed. The set of all vectors $x = (x_1, \cdots, x_m)$ satisfying the inequalities forms a convex set in m-dimer sional space. At an extreme point x_0 of this set, the method characterizes the edges of the set which meet at x_0, and decides, analytically, whether it is possible to move along an edge to increase the functional or whether x_0 is a maximal extreme point. The method is compared with the existing simplex method, and the two methods related geometrically.

2. *Determination of primitive polynomials in a Galois field*, by Dr. E. B. Weinberger, Gulf Research and Development Company.

A Galois field, $GF(p^n)$, is a finite field. Its non-zero elements form a cyclic multiplicative group of order $p^n - 1$. Those elements whose powers generate the group are called *primitive*. Every non-zero element satisfies an algebraic equation of degree n,

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0 \pmod{p}$$

where the a_i are integers mod p, p a prime. The left members of those equations satisfied by primitive elements are called *primitive polynomials*.

Three methods for finding primitive polynomials belonging to a given field were described. By an application of group theory it was then shown that if one primitive polynomial is known all the rest may be found from it. The author then described three more methods of solution based on this process.

3. *A factoring method for differential equations*, by Professor F. H. Steen, Allegheny College.

This paper presented a method leading to the factorization

$$(D^2 - 2aD + a^2 + b^2)y = (D - a - b \tan bx)(D - a + b \tan bx)y$$

which affords a means for obtaining the complementary function of a second order linear differential equation with constant coefficients and imaginary "roots" without resorting to Euler's expression for $e^{i\theta}$.

4. *Concerning metamathematics and Gödel's theorem*, by Mr. F. C. DeSua, University of Pittsburgh, introduced by the Secretary.

An exposition of some methods and results in metamathematics was presented. Results were stated informally without proof although heuristic accounts were given of Gödel's theorem and of Post's proof of the consistency of the propositional calculus.

5. *Some functions related to harmonic functions*, by Professor I. M. Sheffer, The Pennsylvania State College.

Let $u(x, y)$ be an analytic function of the real variables x, y and such that the power series for u and all its derivatives converge in and on a fixed circle C. There exist such functions u that are non-harmonic and such that u and all its derivatives satisfy the Gauss mean value theorem for this one circle C. Properties of these functions are examined. In particular, they satisfy a linear, homogeneous partial differential equation of infinite order in the Laplacian Δu, with constant coefficients.

6. *Exterior forms in Hilbert space*, by Professor T. T. Tanimoto, Allegheny College.

A Grassmann algebra of multilinear mapping in Hilbert space is constructed and a few applications are given.

7. *On finding the characteristic equation of a square matrix*, by Mr. W. Berger and Professor E. A. Saibel, Carnegie Institute of Technology, presented by Mr. Berger.

Using the elementary operations which leave the characteristic equation of the matrix unchanged, it is always possible to transform the matrix such that there is an array of zeros in the lower left-hand corner below the sub-diagonal. Using the Cayley-Hamilton theorem and post-multiplying by the column matrix $\{1, 0, 0, \cdots, 0\}$, a triangular set of equations in the coefficients of the characteristic equation results.

8. *The minimum essentials of a required course in mathematics for business majors*, by Professor Mary A. Goins, Marshall College.

The need for an adequate course in mathematics for students in business curricula prompted this study. A questionnaire containing twenty topics thought to be fairly representative of the minimum content of a reasonably adequate required course in mathematics for business students, with space provided for other suggested topics, was sent to the deans of forty-five professional schools of business in Colorado, Florida, Louisiana, Maryland, Michigan, Oregon, and Texas. The replies were tabulated and analyzed with reference to each curriculum. The final analysis showed that the topics mentioned more than fifty percent of times in all curricula might be taught adequately in a three semester-hour course. If six semester-hours were given to the subject, the topics which were required by all of the schools in all curricula could be taught thoroughly.

F. H. STEEN, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirtieth annual meeting of the Indiana Section of the Mathematical Association of America was held jointly with the Indiana Council of Teachers

of Mathematics at Ball State Teachers College, Muncie, Indiana, on May 1 and 2, 1953. A dinner meeting on Friday evening was addressed by Professor R. E. Langer of the University of Wisconsin on the topic, "Teaching—a job or a profession."

Professor C. P. Sousley, Vice-Chairman of the Indiana Section of the Association, presided at the general session and at the sectional meetings of the Association on Saturday. The general session was devoted to a panel discussion on the academic training of secondary school teachers of mathematics. Professor P. D. Edwards, Head of the Mathematics Department of Ball State Teachers College, acted as moderator, and members of the panel were Professor W. L. Ayres, Dean of the School of Science, Purdue University, Professor Philip Peak of the School of Education, Indiana University, and Mr. R. O. Virts, Chairman of the Mathematics Department and Vice-Principal, Central High School, Fort Wayne, Indiana. The discussion was summed up by Professor R. E. Langer of the University of Wisconsin.

A sectional meeting of the Association devoted to short papers followed the general session in the morning and was continued following the business meeting in the afternoon.

There were one hundred eighteen in attendance including the following thirty-nine members of the Association:

H. W. Alexander, W. L. Ayres, Juna L. Beal, Stanley Bolks, C. F. Brumfiel, W. W. Chambers, K. W. Crain, H. E. Crull, D. E. Deal, M. W. DeJonge, Olive M. Draper, W. E. Edington, P. D. Edwards, S. H. Gould, H. E. H. Greenleaf, Ralph Hull, M. W. Keller, E. L. Klinger, R. E. Langer, Olive Lescow, Florence Long, Gladys B. McColgin, G. T. Miller, Vera T. Morris, Gloria Olive, P. W. Overman, T. P. Palmer, Philip Peak, J. C. Polley, D. H. Porter, Arthur Rosenthal, R. M. Ross, A. R. Schmidt, Sister Gertrude Marie, Aubrey Henderson Smith, C. P. Sousley, Anna K. Suter, R. O. Virts, and M. S. Webster.

The following officers were elected: Chairman, Professor C. P. Sousley, Rose Polytechnic Institute; Vice-Chairman, Professor H. W. Alexander, Earlham College; Secretary-Treasurer, Professor J. C. Polley, Wabash College.

The following papers were presented:

1. *The General Electric fellowship program for secondary school teachers of mathematics*, by Professor Ralph Hull, Purdue University.

The nature and purpose of the General Electric fellowship program, the selection of participants, and plans for its operation on the Purdue campus during the summer of 1953 were described.

2. *Plastic-strip models for non-euclidean geometry*, by Professor H. W. Alexander, Earlham College.

If a narrow, straight strip is cut from thin plastic and laid upon a curved surface in as close contact as possible, it will tend to lie upon a geodesic. If a suitable mesh of geodesics is chosen upon a curved surface, then an approximate model of the surface may be constructed by fastening narrow plastic strips together in a pattern isometric to that on the curved surface. This method is used to fabricate models for non-euclidean geometries of elliptic and hyperbolic type. We thus obtain flexible sections of the sphere and the pseudosphere.

3. *Is undergraduate mathematics part of general education?*, by Sister Gertrude Marie, Marian College.

Discrepancy between general education values of mathematics and the place accorded to mathematics in undergraduate curricula of representative colleges and universities is shown. Resources of mathematics capable of contributing significantly to realization of objectives of general education are balanced against the trend to relegate the subject to the status of "not required" or "alternative with science or philosophy." Recommendations include re-examination of the relationship of mathematics to other disciplines, revaluation of course materials on the basis of general importance, adoption of improved methods and newer instruction media, logical combination of branches of mathematics, and integration of subject matter with the student's total experience.

4. *The School and College Study of Admission with Advanced Standing*, by Professor J. C. Polley, Wabash College.

A brief description of a study, financed by grants from the Fund for the Advancement of Education, in which twelve colleges and twelve secondary schools are collaborating to examine the feasibility of, and set up standards for, granting college credit for courses taken in secondary school. Considered in particular are recommendations of the sub-committee on mathematics concerning a course sequence for the tenth, eleventh, and twelfth years, designed to replace present offerings and cover materials frequently included in the first year of college mathematics.

5. *Visualization in the integral calculus*, by Professor S. H. Gould, Purdue University.

On the principle that "ontogeny recapitulates phylogeny" the speaker discussed the advantage that can be gained in present day classrooms by visualizing the early attempts in the history of mathematics to find the volume of various solids.

6. *Sequential limit spaces*, by Professor J. L. Lawrence, Wabash College, introduced by Professor Polley.

Necessary and sufficient conditions on the collection of open sets are obtained in order that a subset having x as a limit point will contain a sequence of distinct points convergent to x. Such spaces are shown to be completely determined by a knowledge of the convergent sequences.

J. C. POLLEY, *Secretary*

THE MAY MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Louisville, Louisville, Kentucky, on May 9, 1953. Professor W. L. Moore, Chairman of the Section, presided at the morning and afternoon sessions.

Fifty-seven persons were present, including the following thirty-eight members of the Association:

H. H. Berry, J. M. Boswell, M. C. Brown, W. M. Bullitt, Esther A. Compton, J. B. Cornelison, H. H. Downing, R. I. Fields, Clarence Ford, A. W. Goodman, Reverend H. H. Gottbrath, Beulah Graham, Charles Hatfield, Aughtum S. Howard, G. B. Huff, Tadeusz Leser, A. G. McGlasson, D. G. Miller, W. L. Moore, R. S. Park, W. H. Pell, Sallie E. Pence, D. W. Pugsley, V. Elise Qualls, G. G. Roberts, W. J. Robinson, F. E. Ross, J. H. Simester, Sister M. Rosalin, Sister Mary Charlotte, R. H. Sprague, Guy Stevenson, R. P. Tapscott, J. T. Vallandingham, J. A. Ward, R. H. Wilson, T. M. Wright, W. M. Zaring.

At the business meeting the following officers were elected: Chairman, Professor Sallie E. Pence, University of Kentucky; Secretary-Treasurer, Professor W. J. Robinson, Centre College of Kentucky.

The Section unanimously passed a resolution congratulating the present Traveling Speaker, Professor W. L. Moore. Professor Moore announced that he had given addresses in five colleges. The Section unanimously decided to continue the project for next year and accepted the recommendation of the Speaker Committee to elect Professor Aughtum S. Howard of Kentucky Wesleyan College as Traveling Speaker for 1953–54.

By invitation of the Executive Committee Professor G. B. Huff, Head of Department of Mathematics, University of Georgia, delivered the invited address "On Introducing Arguments into Freshman Mathematics."

The following papers were presented:

1. *Center of mass vs. center of gravity*, by Professor R. H. Wilson, Jr., University of Louisville.

This paper is a solution of problem E 1049 of this MONTHLY in which it is requested to find the distance between the center of mass and center of gravity for a thin rod of length L perpendicular to and touching the earth's surface. Dr. Wilson gave that distance to be $(3L^2/8R-5L^3/16R^2)$, where R is the radius of the earth. He then suggested this as an approach to the study of the tides.

2. *On the well-defined principle for groups*, by Mr. R. H. Sprague, University of Kentucky.

Mr. Sprague gave an illustration to show that a group is improperly defined unless its operation is "well-defined" with respect to its equals relation.

3. *A further condensation of Whitehead-Huntington postulate set for Boolean algebra*, by Professor D. G. Miller, University of Louisville.

Dr. Miller showed how to reduce the seven Huntington transformation postulates by two by methods similar to earlier condensations by Bernstein (1916) and Diamond (1934). The complete new postulate set is: P-1, closure for $+$; P-2, closure for $\cdot$; P-3, $[(a+c)(a+b)]d=(bc)d+ad$; P-4, $a=a+bb'=bb'+a=a(b+b')=(b+b')a$; P-5, there are at least two elements. The above set compares in brevity with other short formulations which are in terms of other operations, but is less elegant than some of them because of the complexity of P-4. Equivalence to Diamond's set was indicated. Consistency and independence examples can be exhibited.

4. *Invariants of G.C.D. algorithms*, by Professor N. G. Bruijn, Muiderberg, Netherlands, and Mr. W. M. Zaring, University of Kentucky, presented by Mr. Zaring.

Mr. Zaring outlined a method of constructing sequences of algorithms which, when used to relate an arbitrary greatest common divisor algorithm to the least remainder algorithm, or Euclid's algorithm, establishes properties which are invariant for each element of the sequence. Hence, these properties are invariant for all G.C.D. algorithms for the same two positive integers.

5. *Some mathematical hurdles*, by Sister M. Matilda, Loretta Junior College, introduced by the Secretary.

Some mathematical hurdles based on practical experience in grade and high school mathematics were treated in this paper, and simple and effective solutions were suggested. These showed that most mathematical difficulties can be remedied by an enthusiastic, resourceful, and well prepared teacher. Ingenuity, patience, perseverence, and above all, love of those taught will bring success to both the teacher of mathematics and to her pupils.

6. *Some first editions of famous mathematicians*, by Mr. W. M. Bullitt, Louisville, Kentucky.

Mr. Bullitt gave a very interesting account of the history of several first editions of famous mathematical papers in his collection. Since the originals are now on display at Harvard, he showed photographs of them.

7. *What should we teach in freshman mathematics*? A panel discussion with the following members: President J. M. Boswell, Cumberland College; Mr. J. W. Huss, auditor with Lybrand-Ross Bros., and Montgomery; Professor W. H. Pell, University of Kentucky; Professor W. J. Robinson, Centre College; Professor J. H. Simester, Speed Scientific School, University of Louisville.

The members of the panel deplored the high school preparation of freshmen and gave various methods of meeting that difficulty. Each member of the panel showed how he handled the problem.

8. *On introducing arguments into freshman mathematics*, by Professor G. B. Huff, University of Georgia. (By invitation.)

Professor Huff began by suggesting that the widespread dissatisfaction with freshman mathematics may be relieved by introducing mathematical arguments at every opportunity. If the fundamental laws and the number systems are mentioned early and by name, it was shown that arguments may be made in connection with most of the usual subject matter. In this way freshmen have the opportunity to do mathematics themselves and to draw logical conclusions from the results of algebraic operations.

9. *Gauss-Lucas theorem*, by Professor A. W. Goodman, University of Kentucky.

An expository talk covering the Gauss-Lucas theorem and Jensen's extension.

10. *Elementary methods of forecasting as used in a specific utility rate case*, by Mr. F. E. Ross, Lybrand-Ross Bros., and Montgomery.

The forecasts were made in the hearings of a state regulatory agency in the application of a city bus company for increased fares. Historical data as to actual riding experience were normalized to adjust for strikes and forecasts were made by methods of finite differences, least squares, and more simple percentage relationships. Sampling methods were not considered as feasible or conclusive for such forecasts.

J. A. WARD, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the United States Naval Proving Ground, Dahlgren, Virginia, on May 2, 1953. Professor Marian M. Torrey, Chairman of the Section, presided at the morning and afternoon sessions.

There were one hundred eleven persons in attendance, including the following sixty-five members of the Association:

J. C. Abbott, D. F. Atkins, R. P. Bailey, N. H. Ball, J. E. Barker, W. E. Barnes, J. W. Blincoe, T. A. Botts, C. C. Bramble, Mary P. Burkhart, H. H. Campaigne, J. F. Canu, S. H.

Chasen, G. R. Clements, D. F. Eliezer, W. L. Fields, A. M. Fleishman, C. H. Frick, G. H. Gleissner, Michael Goldberg, R. A. Good, E. S. Grable, E. C. Gras, J. J. Greever, III, J. L. Hammersmith, J. R. Hammond, Bertha I. Hart, M. Gweneth Humphreys, W. R. Hydeman, R. D. Johnson, L. M. Kells, F. B. Key, H. L. Kinsolving, V. L. Klee, Jr., Karl Kozarsky, D. C. Lewis, Jr., T. J. Little, D. B. Lloyd, E. S. Mayer, Carol V. McCamman, E. J. McShane, Joseph Milkman, P. L. Oglesby, J. F. Paydon, J. W. Popow, J. D. Riley, E. K. Ritter, R. E. Root, E. D. Schell, Veryl G. Schult, T. McN. Simpson, Jr., M. F. Stilwell, H. C. Stotz, W. J. Strange, Choy-tak Taam, J. A. Tierney, John Todd, Marian M. Torrey, John Tyler, Mary C. Varnhorn, C. H. Wheeler, III, B. F. Whisler, J. W. Wrench, Jr., W. W. Youden, D. M. Young, Jr.

The following officers were elected to serve for a period of one year: Chairman, Professor D. C. Lewis, Jr., The Johns Hopkins University; Vice-Chairmen, Professor R. P. Bailey, United States Naval Academy, and Professor E. S. Grable, University of Richmond; Secretary, Professor C. H. Frick, Mary Washington College of the University of Virginia. A report from Mr. W. H. Norris, Chairman of the High School Contest Committee, was read by the secretary. Mr. Norris expects to be able to report on the number of high schools willing to take part at the December meeting.

The following papers were presented:

1. *On the expansion of a function in a continued fraction by means of confluent reciprocal differences*, by Mr. J. E. Barker, United States Naval Proving Ground.

The relation between the Taylor's series of a function and the successive convergents of Thiel's continued fraction was shown. These convergents are generating functions of recurring series which coincide with Taylor's series up to terms of a certain order. Numerical examples were given, including expansions of a trigonometric and an exponential function which began deviating from the Taylor's series at the fifth degree terms. These expansions give good results in the numerical evaluation of non-integral functions.

2. *The condition of a matrix*, by Mr. John Todd, Computation Laboratory, National Bureau of Standards.

A three by three matrix was given in which a change in the value of one element by a factor of one tenth of one per cent changes the value of the determinant of the matrix by a factor of more than one hundred. Similar situations can arise in the solutions of certain systems of linear equations and the inverses of certain matrices. These phenomena are ascribed to the "lack of condition" of the matrix. Various measures for this were indicated. It was shown that, although the matrices which arise in practice are often in good condition, some innocent looking matrices are incredibly bad.

3. *Jump mechanics*, by Professor John Tyler, United States Naval Academy.

If a curve is defined by the equations $x=x(t)$, $y=y(t)$, the process of replacing t by $t+h$ will give a jump from the t point to the $t+h$ point of the curve. For certain curves $x(t+h)$ and $y(t+h)$ can easily be expressed as functions of $x(t)$ and $y(t)$. The resulting equations are called the "jump equations" of the curve. Jump equations were given for the parabola, the circle, the ellipse, and the exponential spiral. Linear "jump equations" can be found for any curve for which $x(t)$ and $y(t)$ are sums of exponentials in t or for any mechanical system whose differential equation, $f(y, y', y'', \cdots, x)=0$, is a linear differential equation with constant coefficients.

4. *Real-valued continuous functions defined on spheres*, by Mr. R. D. Johnson,

University of Virginia.

If f is real-valued and continuous on a 2-sphere, S^2, there exist three points p_1, p_2, p_3 on S^2 such that

$$f(p_1) = f(p_2) = f(p_3),$$

and there exists a point r contained in $f(S^2)$ such that some component of $f^{-1}(r)$ contains a pair of antipodal points of S^2. These results give a generalization of a result obtained by Yamabe Yujobo, and include as a special case two related theorems of Kakutani and de Mira Fenandes. The theorems proved yield corollaries pertaining to properties of a two closed set covering of S^2.

5. *The logical design of the Dahlgren calculators*, by Mr. R. A. Niemann, United States Naval Proving Ground, introduced by the Chairman.

The designs of the Aiken Relay Calculator (Mark II) and of the Mark III Electronic Calculator were discussed. The storage capacity of the Mark II is 100 ten digit numbers, that of the Mark III is 4350 sixteen digit numbers. The output of the Mark II can be recorded on punched paper tape or printed directly, that of the Mark III is given on magnetic tape which can be fed into electric typewriters for printing. The Mark II can do twenty-four operations per second, the Mark III can do one hundred forty. Trends in the design of future computers were discussed.

6. *Certain problems of approximation using Bernoulli and Euler polynomials*, by Professor D. C. Lewis, Jr., The Johns Hopkins University.

This was the invited lecture. The problem of expanding a function in terms of its even order derivatives at two distinct points leads to series of Bernoulli polynomials. To expand it in terms of even derivatives at one point and odd derivatives at the other point leads to series of Euler polynomials. Such expansions are possible only for entire functions of exponential type, the l.u.b. of the permitted type being inversely proportional to the distance between the two points. Results of this kind were obtained in part by H. Poritsky and N. Bielson. Certain determinant formulas were obtained for both Bernoulli and Euler polynomials.

C. H. FRICK, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The May meeting of the Minnesota Section of the Mathematical Association of America was held at St. Olaf College in Northfield, Minnesota, on May 9, 1953. Sessions were held in the forenoon, at luncheon and in the afternoon. Professor C. S. Carlson, Mr. J. S. Hill and Professor A. G. Swanson, Chairman of the Section, presided at the respective sessions.

Seventy-two persons attended the meeting including the following fifty-two members of the Association:

H. M. Anderson, W. P. Anderson, J. J. Bailey, E. J. Berger, K. H. Bracewell, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson, Elizabeth Carlson, Helen Engebretson, D. H. Firl, Walter Fleming, Abraham Franck, W. B. Fulks, Gladys Gibbens, J. E. Hafstrom, Charles Hatfield, Jr., F. C. Hatfield, A. G. Hill, J. S. Hill, Hildegarde H. Howden, D. A. Johnson, N. W. Johnson, Jr., Rev. W. C. Kalinowski, G. K. Kalisch, Karlis Kaufmanis, M. L. Keedy, W. S. Loud, Walter Lyche, H. B. MacDougal, K. O. May, W. R. McEwen, E. D. Nering, M. J. Norris, P. C. Rosenbloom, Ruth Scholten, Sister Mary Leontius, Sister Mary Seraphim, F. C. Smith, O. E. Stanaitis, A. G. Swanson, F. J. Taylor, Takashi Terami, Tung Tsang, O. E. Walder, Frances E. Walsh, C. Y. Wang, K. W. Wegner, Irene L. Wente, R. P. Winter, R. S. Wollan.

At the business meeting, the following officers were elected for the coming

year: Chairman, Professor A. G. Hill, North Dakota State College; Secretary, Professor F. C. Smith, College of St. Thomas; Executive Committee, Professor W. S. Loud, University of Minnesota, Sister M. Leontius, College of St. Teresa, Professor A. G. Swanson, Gustavus Adolphus College.

By invitation of the Executive Committee, Professor W. S. Loud of the University of Minnesota delivered an address at the morning session entitled "Some Examples of Physical Reasoning in Mathematical Research." Abstract of this address follows:

Much important research in mathematics is physically motivated. A useful technique in such areas is to consider a physical model of the mathematical situation and use its properties to suggest directions of mathematical investigation. Strict proof of mathematical properties is still required. Several illustrations from non-linear problems were presented as illustrations of this idea.

The following short papers were presented:

1. *Condensed graph paper for the cartesian plane*, by Professor K. O. May, Carleton College.

The graph paper maps the plane on the interior of a circle with center at the origin by means of two projections, the first onto the lower half of a sphere tangent to the plane at the origin by projection to its center, the second orthogonally onto the plane. The coordinate grid consists of two families of ellipses with major axes equal to the diameter of the circle. Straight lines go into ellipses, circles with center at the origin into circles of the same kind, and straight lines through the origin into segments of themselves. Distortion is small near the origin and symmetries about the origin and any line through it are preserved. Unbounded curves are easily graphed and their asymptotic or other behavior "at infinity" visualized.

2. *The asymptotic evaluation of an integral depending on two large parameters*, by Mr. R. N. Pederson, University of Minnesota, introduced by Professor W. B. Fulks.

The author considers integrals of the form

$$I(h, k) = \int_0^a f(t) \exp\left[-h\phi(t) + k\psi(t)\right]dt,$$

where $\phi(t)$ has a fourth order minimum at $t=0$ (*i.e.*, $\phi'(0)=\phi''(0)=\phi'''(0)=0$, $\phi^{iv}(0)>0$), the second order case having previously been discussed. Estimates were given for the value for large h and k in all cases where at least one of the first four derivatives of $\psi(t)$ is non-vanishing at the origin, $f(0)\neq 0$, and $k=o(h)$.

3. *The mathematical concept of a mechanical system*, by Professor P. C. Rosenbloom, University of Minnesota.

We define a mechanical system with n degrees of freedom as a smooth manifold of $2n$ dimensions such that: (1) in each local coordinate system there is a distinguished pairing of coordinates (q_i, p_i), $(i=1, 2, \cdots, n)$; (2) the transformation generated by overlapping coordinate patches leaves the exterior differential form $\sum dq_i dp_i$ invariant; (3) there is a distinguished scalar H of class C^2; (4) the solution of the equations $dq_i/dt=\partial H/\partial p_i$, $dp_i/dt=-\partial H/\partial q_i$ with arbitrary initial values (q_i^0, p_i^0) is defined for all $t\geqq 0$. With this formulation, the assertions of theoretical mechanics become perfectly good theorems in the differential geometry of a class of smooth manifolds based on the local group of canonical transformations. For most purposes of statistical mechanics, a simpler structure is sufficient. An energy system is a space S with a measure V such that $V(S)=+\infty$ and a non-negative measurable function H such that the measure $V_H(x)$ of the set where $H\leq x$

is finite for all x such that $V<+\infty$, and such that $\lim_{x\to\infty}\log V(x)/x=0$. This formulation applies to quantum as well as to classical mechanics.

4. *A diagrammatic interpretation of the formal definition of the derivative of* $f(x)$, by Professor A. L. Duquette, St. John's University, introduced by Reverend W. C. Kalinowski.

Given any ϵ, one can construct an ϵ-neighborhood containing the endpoint of a line segment representing the derivative $f'(x_0)$, such that for a corresponding δ-neighborhood about x_0, all of the chords in the δ-neighborhood passing through x_0, when extended, pass through the ϵ-neighborhood.

5. *On the sum of the face angles and the sum of the angular deficiencies of a polyhedron*, by Mr. N. W. Johnson, Jr., Carleton College.

In terms of a new definition of the density d of a polygon, it is found that the sum of the exterior angles of any polygon is $2\pi d$. The sum of the face angles of a polyhedron of genus p with V vertices and f_d faces of density d is given by

$$\sum\beta = 2\pi\left[V-2+2p-\sum_d f_d(d-1)\right].$$

The amount δ by which 2π exceeds the sum of the face angles at a vertex of a polyhedron is defined to be the angular deficiency at that vertex. It is shown that

$$\sum\delta = 2\pi\left[2-2p+\sum_d f_d(d-1)\right].$$

Examples are given to illustrate both formulas.

6. *Report on the two latest electronic computers: the IBM 701 and the ERA 1103*, by Mr. J. Stanley Hill, Actuary, Minnesota Mutual Life Insurance Company.

A very brief description of each machine was given and their likenesses and dissimilarities were noted. The remarkable improvement in speeds over earlier electronic calculators was noted. Emphasis was placed upon the fact that these are "scientific" calculators, capable of handling problems with tremendous calculational magnitude, rather than "commercial" calculators whose ratio of calculating capacity to input and output capacity is much lower.

F. C. SMITH, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The ninth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the United States Military Academy, West Point, New York, on May 8 and 9, 1953. An informal reception was held on May 8; on May 9 the Chairman of the Section, Professor J. F. Randolph of the University of Rochester, presided at the morning session, and the Vice-Chairman, Professor Harriet F. Montague of the University of Buffalo, presided at the afternoon session.

One hundred three persons attended the meeting, including the following fifty-nine members of the Association:

R. A. Beaver, R. L. Beinert, W. W. Bessell, L. H. Bunyan, F. J. H. Burkett, E. A. Butler, W. B. Carver, Mary P. Dolciani, E. J. Downie, Howard Eves, A. B. Farnell, O. J. Farrell, Jean Feidner, J. E. Freund, E. I. Gale, H. M. Gehman, B. H. Gere, Lillian Gough, N. G. Gunderson,

Anna M. Howe, G. W. Kays, J. R. F. Kent, F. W. Lane, Violet H. Larney, R. D. Larsson, Caroline A. Lester, J. V. Limpert, R. C. Luippold, C. T. Male, J. D. Matheson, June M. McArtney, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, Anne Motycka, C. W. Munshower, C. V. Newsom, Abba V. Newton, C. P. Nicholas, F. D. Parker, Theresa L. Podmele, J. F. Randolph, Audrey I. Richards, Charles Roth, Edith R. Schneckenburger, A. T. Skinner, E. P. Starke, Ruth W. Stokes, Irwin Stoner, Mary C. Suffa, R. L. Swain, Nura D. Turner, J. P. Van Alystyne, S. I. Vrooman, J. F. Wardwell, Mary E. Williams, A. G. Wootton, Frances M. Wright, R. C. Yates.

At the business meeting the following officers were elected: Chairman, Professor Harriet F. Montague, University of Buffalo; Vice-Chairman, Professor J. R. F. Kent, Harpur College; Secretary, Professor N. G. Gunderson, University of Rochester.

The following papers were presented:

1. *The teaching of mathematics at West Point*, by Colonel W. W. Bessell, United States Military Academy.

Using charts and a military analogy—artillery fire—Colonel Bessell covered: incoming cadets (projectiles), motivation for a lifetime military career (initial and booster charges), teachers and teaching methods (gun, flight controls), course of instruction (trajectory), and objective (target) of the Department of Mathematics at West Point. Procedures peculiar to USMA included: control of study time and habits; accent on development of character (responsibility) and power of analysis; supplementing texts with military applications; daily recitations and grades; small (15 cadet) sections with cadets arranged in order of merit; annual replacement of one third of officer instructors; courses for instructors in teaching methods and higher mathematics.

2. *Color and mathematics*, by Miss Nura D. Turner, State University of New York, College for Teachers at Albany.

The speaker discussed the solution of three simultaneous linear equations by a graphical method which illustrates the practical and usual procedures in a certain industrial plant to determine the percent of each component of a three component dyestuff mixture necessary to produce a spectral match of that mixture. The use of a spectrophotometer equipped with an automatic tristimulus integrator to obtain the numerical values of the constants and coefficients of the variables was described and explained.

3. *The conicograph*, by Mr. E. I. Gale, Norland, Ontario, Canada.

A novel motor-driven conicograph was exhibited in which Peaucellier's ellipsograph attached to a revolving plane, imparted "camwise" rectilinear motion through a chain of linkages to the opposite side where the conic was drawn. Simple means of making inversors dynamically balanced and free from "dead" points was shown. A linkage combination was demonstrated which described simultaneously both branches of the equiangular hyperbola, using the inverse of Bernoulli's lemniscate. A unilateral pantograph efficiently transferred the conics to the blackboard using the principle of "fore-shortening."

4. *Conic sections and other curves on the screen and in space*, by Professor Ruth W. Stokes, Syracuse University.

Professor Stokes gave a brief discussion of conic sections as defined by the ancient Greeks. Using string models of cones and a plane of light from the projector, the speaker showed the conic sections. Also, by use of a string model of the hyperbolic paraboloid and a plane of light, parabolic and hyperbolic sections were shown. Finally, by intercepting a curved "surface" of light in space by a string model, space curves were illustrated.

5. *The Mangler transformation*, by Mr. Irwin Stoner, Bell Aircraft Corporation.

The speaker derived the transformation of the flow equations about bodies of revolution to the flow equations about 2-dimensional bodies which was stated by Mangler in 1945 and 1946. The derivation considered the equations of motion written in terms of the stream function. The special cases of a conical body and an ogive were then considered.

6. *Teaching a course in the history of elementary mathematics*, by Professor Howard Eves, Champlain College.

In teaching a course in History of Elementary Mathematics the speaker wished to make the students' participation something more than carrying out reading assignments capped with a term paper. A solution was found by forming a catalogue of problem studies, arranged to fit the lecture sequence. Weekly assignments from these problem studies serve to stimulate student interest, give a concrete aspect to the course, crystallize many historical methods and processes, furnish valuable classroom material for the future teacher, and introduce the mathematics major to a number of things of mathematical interest often not encountered in his other courses. This paper concerned itself with an explanation and discussion of these problem studies.

N. G. GUNDERSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 5, 1953.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, April, 1954.

MINNESOTA, Bemidji State Teachers College, October 10, 1953.

MISSOURI, University of Missouri, Columbia, Spring, 1954.

NEBRASKA

NORTHERN CALIFORNIA

OHIO, April, 1954.

OKLAHOMA, Oklahoma City University, October 30, 1953.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.

ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April, 1954.

SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.

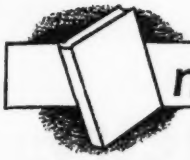
SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.

SOUTHWESTERN, Arizona State College, Tempe.

TEXAS, Texas Technological College, Lubbock, April, 1954.

UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.

WISCONSIN, State Teachers College, Eau Claire, May, 1954.



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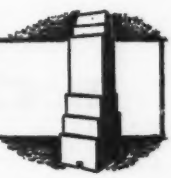

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